The Schrödinger-Newton Equations

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Abstract

This essay explains the motivation behind the Schrödinger-Newton equations, and offers a mathematical overview of them, including a calculation of their Lie point symmetries.

Contents

1 Gravity and the quantum world

1.1 Introduction

The Schrödinger-Newton (or 'SN') equations result from the introduction of a gravitational self-interaction potential term into the Schrödinger equation of quantum mechanics. Although the strength of gravity is negligible on the quantum scale, particularly in the face of the electromagnetic, strong and weak fundamental forces, it has been proposed by many people over the years that gravity may be a mechanism by which the mysterious process of 'wavefunction collapse' occurs, due to the apparent instability created by a superposition of two different space-times.

The SN equations were proposed by Penrose [7, 8] to help shed light on this problem, although the same system has been studied before as the 'Gravitational Schrödinger Equation' [9] (which is not to be confused with the standard Schrödinger equation using a modified Planck's constant [22]) and fairly extensively as the 'Schrödinger-Poisson system' [6, 11].

He believes that matter can 'objectively' collapse into a single state through this instability, which is why we see macroscopic objects to be in just one state at a time, and that the SN equations give the basic states that will not collapse any further. However, they are not the beginnings of some 'Theory of Everything' that unifies general relativity and quantum mechanics, which is one of the biggest aims of modern physics. Penrose's Twistor theory works in this direction, and he holds the belief that "... It is with gravitational effects that the very rules of quantum theory (and quantum field theory) will have to undergo profound modification." [13]

1.2 Quantum state reduction

The wavefunction of a particle or system mathematically, at least, represents a superposition of all its possible states. We can use the Schrödinger equation to see how it evolves unitarily over time; that is, given an initial state $|\Psi(0)\rangle$, we can calculate future states using an operator $\hat{\mathcal{U}}$:

$$|\Psi(t)\rangle = \hat{\mathcal{U}}(t)\,|\Psi(0)\rangle\,. \tag{1}$$

It is easy to show that evolution under a time-independent Hamiltonian $\hat{H}$ is given by

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\Psi(0)\rangle \,. \tag{2}$$

From the state vector $|\Psi\rangle$ we are able to calculate the probability that a particle or system will be in a particular position, energy or some other eigenstate at a given time when it is 'observed.'

This act of observation – some measurement that we make of the system – appears to instantaneously 'collapse' the wavefunction, because we only observe a single and definite eigenstate out of the many (possibly infinite) that are possible. The experimental 'evidence' for this is quite strong; see, for example, the wave-particle duality exhibited by electrons in the double-slit experiment.[1] This is a process known as wavefunction collapse, quantum state reduction or collapse of the state vector. We will furthermore refer to it by **R**, echoing Penrose's notation (first used in [7]), and to unitary wavefunction evolution by **U**.

Some interpretations of quantum mechanics take the view that **R** does not actually happen, or is simply a part of **U**, but others consider the two to be completely different processes. This is mainly because **U** deals with the evolution of a superposition of states in a time-symmetric process, rather than just an instantaneous collapse into a single eigenstate. If we take **R** to occur, we need to answer several questions: how is the new state chosen from those that are possible (and is such a selection random), why does it occur instantaneously, and what counts as an 'observation' that triggers the collapse?

This is generally referred to as the measurement problem. It is famously illustrated by a thought experiment published in 1935 by Erwin Schrödinger, known as 'Schrödinger's cat,' which was to show that quantum mechanics could not describe the observed 'definite' behaviour of macroscopic objects. This setup has a cat placed inside a sealed box, along with some mechanism that

[1] A famous experiment, first performed by Thomas Young around 1805, whereby beams of photons, electrons or even molecules are fired through two or more thin slits in an opaque wall, and are observed creating interference patterns behind – something which does not happen with only one slit open. This is consistent with the idea that the beam is a wave which diffracts and interferes with itself upon passing through the slits, and is still seen if we were to only fire a handful of particles, or only one, through the slits! At the time of writing, this experiment has reportedly been carried out with the double slits in *time* instead of space, with similar results [26]

has a probability of killing the cat in a given time frame: for example, a single atom of ^{249}Fm, which has a half-life of 3 minutes, near a Geiger counter that will cause a vial of poison gas to be broken upon detection of any decay. Our unstable atom is, for all intents and purposes, in a superposition of being both decayed and un-decayed until it is observed, and so the question is whether or not the cat is also entangled[2] with the nucleus – is it dead, alive, or in a superposition of being both dead and alive at the same time until we open the box? Like the radioactive atom, the cat should be in a state

$$|\Psi_{\text{cat}}\rangle = \alpha(t)\,|\text{dead}\rangle + \beta(t)\,|\text{alive}\rangle \tag{3}$$

until the box is opened (where the ratio $|\alpha(t)|^2 : |\beta(t)|^2$ gives the probabilities of the cat being dead/alive at time t, and we should expect α to become larger relative to β over time in this situation, and the basis states are orthogonal; that is, $\langle\text{dead}|\text{alive}\rangle = 0$).

The other question is what an 'observation' actually is – can the Geiger counter itself collapse the superposition, or will the cat do it, or does it require a human being? What happens if a friend of yours looks inside but doesn't tell you what's happened to the poor cat – does he become entangled too and fail to collapse the cat's state until you yourself check? (This is another famous thought experiment, known as 'Wigner's friend.') And do we require some sort of 'god' or 'supreme consciousness' to be observing the Universe, and allow us to evolve from superimposed strings of primordial DNA into observers ourselves?

One should also raise the topic of 'un-collapsing' the wavefunction. Upon placing Schrödinger's cat into its box it must initially be in an observed, definite state, but then when the lid is closed we are taken to a superimposed set of states again instantly. Is there an equivalent, reversed form of **R**, that goes from a definite state into a superposition? Surely, if there wasn't, then much of our Universe should be expected to have become entangled with itself by now,

[2]There is another feature of quantum mechanics called 'entanglement,' whereby two or more interacting, unobserved particles will be 'linked together' and can almost be thought of as the same entity. The feature that we are interested in here is that collapsing the wavefunction of one particle will also instantaneously collapse those of any other particles that are entangled with it, regardless of distance between them (Einstein referred to this as "spooky action at a distance") though one often hears about electron spins becoming entangled.

and thus an observation would collapse the whole thing. However, this reversal process is generally accepted to be part of **U**, as we are not permanently destroying the wavefunction and a superposition of states can again evolve unitarily from the point that we stop observing (although the probability of 'far-away' states will be small soon after observation). But as we shall see, this idea may pose some problems for a theory of objective wavefunction collapse.

Penrose gives an argument as to why **R** can't be time symmetric in [24], via a simple standard 'laser and beam splitter' setup inside a corridor. If one fires high energy x-ray photons at a half-silvered mirror angled at 45° to the beam, then the probability of a photon being transmitted through the mirror and reaching a detector behind it is 50%, and 50% that it will be reflected into a wall. (Note that detecting the photon must cause the wavefunction to collapse since we know which way it's come – even if it doesn't reach the detector we know it's ended up hitting the wall, and will thus still collapse its state in what's called a 'null measurement.') However, if we think about the problem in reverse and consider where the photons came from, by the same logic there is a 50% chance that the photon was initially emitted from the other wall and reflected off the mirror – but one should think that the probability of an x-ray photon spontaneously emitting itself from a wall of this corridor and reaching the detector is close to 0! One could try to counter this argument by saying something about the laws of thermodynamics, but the point made here is that "quantum probabilities for future predictions arising in the measurement process do not seem to depend at all on considerations of particular temperatures or geometries or anything" [24] – essentially, that it's *easier* to predict the future than the past in quantum mechanics, and that **R** must have something to do with it.

And finally, as macroscopic objects are generally not seen to be in these superpositions, can one assume that mass or size scales must have something to do with collapse? But there *are* large systems which exhibit this type of quantum behaviour, such as Bose-Einstein condensates and superconductive materials, which involve a massive number of particles with their wavefunctions 'combined.' The answer, if the question is valid, will probably be due to system complexity, certain 'special' mass distributions, or consciousness itself. Various interpretations of quantum mechanics have been made over the year to explain all these strange experimental results, and all (must) address the measurement problem in some way.

1.3 Interpretations and modifications of quantum mechanics

(Author's note: although we point out problems with some of the various interpretations below, this paper is not biased to believe one of them more than any other, and only wishes to present the possibility of gravity causing objective wavefunction collapse.)

The Copenhagen interpretation takes the view that our 'knowledge' of the system is incomplete. Since we can only attribute values to observables of a system if we measure it, talking about such things only makes sense in the context of a measurement – the observed particles become entangled with the observing apparatus, and so there is no 'reality' outside of such a framework – and **R** is considered to be a 'jump' in our knowledge of the system. Hence, Schrödinger's cat is considered to be both dead and alive simultaneously until such a time that asking if it is either makes sense. However this view does not tell us *when* the collapsing act of observation actually occurs, nor how a random choice is selected. Many people further believe that consciousness is somehow necessary in causing state reduction, due to this idea of 'knowledge' jumping around, but perhaps also because they dread to think of such a thing happening to a cat (it is only horrible little bacteria and viruses that could possibly suffer such a fate!).

The Ghirardi-Rimini-Weber (GRW) modification [2] takes the view that there is a tiny probability that any superposition will collapse spontaneously, and so an object made of some very large number of particles is very unlikely to be seen in such a superposition – in fact, such superpositions are forbidden under this theory, and classical mechanics still results from such behaviour.

A popular view among many physicists today is the idea of parallel universes (also known as Everett's Many-Worlds Interpretation, as he was the first to publish such an idea). This states that all of the possible outcomes of collapse do occur, but in separate universes which are somehow superimposed or 'parallel' to each other, so that all evolution remains unitary (and wavefunctions do not interact with each other between such universes). It also explains the anthropic principle to some degree,[3] but the problem is that (as yet) no feasible ways of testing this have yet been formulated, and its possible that there is no way to. One obvious objection is the conservation of energy involved in creating these new universes, but the theory deals with a superposition of universes all existing at the same time, rather than physically distinct ones, so violation of conservation laws isn't considered to be an issue (if one thinks about it in this way, there is a much more pressing problem: how to instantaneously create a Universe!).

And finally, some (including Albert Einstein, as set out in his joint 1935 'EPR' paper entitled "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?") believe that there are 'hidden variables' involved – indeed, this is the natural underlying idea that we, living in the 'classical' mechanical world, would be inclined to believe. These hidden properties that we can't or haven't measured would infer that quantum mechanical processes are actually deterministic, but for all practical purposes would appear random to us. The most famous version of this is Bohmian mechanics [1]; unfortunately, we'd have to know the exact configuration of particles and measure these hidden variables to be able to predict anything, and given the difficulty of the task for even a handful of objects we're not really placed in a situation that helps us more than a statistical treatment! Is it also claimed that Bell's Theorem[4] shows hidden-variable theory to be invalid, unless it is 'non-local' and violates special relativity, although some supporters claim there are holes in this logic, whilst others work on non-local forms of the theory.

In recent years there has also been work in a field called quantum decoherence, which tries to explain why macroscopic objects aren't observed in superpositions (at the time of writing, the largest object to have displayed wave-particle duality is a molecule of $C_{60}F_{48}$ [21]). Essentially, when one system interacts with another much more complicated 'environmental' system, it is rapidly forced to choose a random state chosen from a 'density matrix.' It is even possible that the Universe can only handle so much complexity at once, through limits on its computational power. Though not an interpretation in

[3]Put simply, that the incredible coincidences that apparently have led to our existence happened because they could do, and we just happen to live in that parallel universe!

[4]This topic itself is worth a whole essay, or two; see, for example, John Bell's original paper *On the Einstein Podolsky Rosen paradox*, Physics 1 (1964) 195-200

its own right, environmental decoherence does assume **R** to occur and shares some ground with the Copenhagen interpretation (in that the observer and his complex measuring apparatus causes the collapse), and offers time scales for such a process to occur. Proponents point out that this is a framework which is a consequence of quantum mechanics, rather than a modification of it, and is hence more readily accepted.

It should be noted that a recent experiment by S. Afshar seems to indicate that the wavefunction does not actually collapse [23]. He fires laser light through two thin slits, which are focused by a lens onto two detectors, each one only picking up light coming from one slit. By placing a wire grille in front of the lens so that the wires lie in the dark parts of the interference pattern created by the beams interacting with each other, the detectors receive the same signal intensity as they would without a grille. The claim is that this shows light behaving as both a wave (interference) and a particle (detection after travelling along just one path) at the same time, thus violating Bohr's principle of complementarity. A strong rebuttal of these results has not yet been made, but experiment was performed quite recently; it is claimed that this result lends support to the Transactional interpretation [3], which involves 'advanced' and 'retarded' waves in time as part of the wavefunction.

Penrose has suggested that gravity is a mechanism by which state reduction occurs [7], and furthermore that it is objective (which he refers to as **OR** for "objective reduction," i.e. it does not require any observer or other environmental effect). Furthermore, gravity would be the *only* method by which **OR** could happen, and would supplement 'normal' **R** occurring through environmental decoherence.

As already stated, he strongly believes that the rules of quantum mechanics need to be changed – although a wildly successful 'theory,' it does not merge well with general relativity and has many problems itself – and he says that, "The quantum measurement problem is, to me, an even clearer reason for requiring a satisfactory union of the ideas of quantum theory and general relativity" [13]. Several of his papers and books gives his objections, but in particular he worries about what happens to information falling into black holes and the infinities present in quantum field theory (see especially [24], chapter 30), along with those presented in the next section.

In fact, he considers consciousness itself to be a result of this occurring within microtubules inside brain cells [5], and believes we perceive time to 'pass' at the rate at which **OR** occurs in these structures. One resulting prediction is that general anaesthetics work by interfering with the microtubules (the mechanism by which they render someone unconscious is not yet known), along with the possibility of truly artificial intelligence. Unfortunately this theory hasn't been developed far enough to explain why, for example, our arm isn't conscious, as it contains similar microtubules (although it should be pointed out that we don't actually know that our arm doesn't have a conscious separate to the one we experience in our head!).

1.4 Penrose's thought experiment

In [7] he introduces a thought experiment involving moving a significant mass (which he names "Schrödinger's lump"), and argues that a superposition of states with different mass distributions – for example, a cat either sitting down or lying dead in Schrödingers' infamous box, or said lump being in one of two different places as a result of some quantum mechanical process – is "ill-defined" with a calculable "energy uncertainty," which leads us to a plausible time-scale in which **OR** will occur.

He considers the stationary states for two different mass distributions, described by their wavefunctions $|\chi\rangle$ and $|\varphi\rangle$, both with the energy eigenvalue E:

$$i\hbar\frac{\partial\,|\chi\rangle}{\partial t} = E\,|\chi\rangle\,, \quad i\hbar\frac{\partial\,|\varphi\rangle}{\partial t} = E\,|\varphi\rangle\,. \tag{4}$$

A linear superposition of states

$$|\Psi\rangle = \alpha\,|\chi\rangle + \beta\,|\varphi\rangle \tag{5}$$

for some constants α, β must therefore have the same energy eigenvalue and be stationary too:

$$i\hbar\frac{\partial\,|\Psi\rangle}{\partial t} = E\,|\Psi\rangle\,. \tag{6}$$

However, general relativity tells us that these masses will create curvature in the local space-times: is the operator $\frac{\partial}{\partial t}$ the same for both mass positions, and is such a superposed state of space-times still a stationary state? In

[7] he points out that there is a certain gauge freedom in general relativity that prevents us from identifying the position of one space-time with another, which is something that we need to do to meaningfully distinguish one mass location from another: an "approximate pointwise identification" is necessary in quantum mechanics, and so we shall assume we can somehow make one.

In a stationary space-time, such as on the surface of the Earth (or even at the centre), Penrose describes that it is "characterized by the existence of a timelike Killing vector $\mathbf{K}$" [24], which we can use to replace the operator $\frac{\partial}{\partial t}$ in (6):

$$i\hbar\mathbf{K}\left|\Psi\right\rangle = E\left|\Psi\right\rangle . \tag{7}$$

So, if we simply place both states on the Earth's surface (at the same height, so that they still have the same energy) we can write

$$i\hbar\mathbf{K}\left|\chi\right\rangle = E\left|\chi\right\rangle , \quad i\hbar\mathbf{K}\left|\varphi\right\rangle = E\left|\varphi\right\rangle \tag{8}$$

which still has a stationary superposition given by

$$i\hbar\mathbf{K}\left|\Psi\right\rangle = E\left|\Psi\right\rangle . \tag{9}$$

However, we need to take into account the gravitational fields of each of the states; if they are both to be stationary, and forming different space-times, then they must actually have *different* associated Killing vectors

$$i\hbar\mathbf{K}_\chi\left|\chi\right\rangle = E\left|\chi\right\rangle , \quad i\hbar\mathbf{K}_\varphi\left|\varphi\right\rangle = E\left|\varphi\right\rangle , \tag{10}$$

and if we try to write down a superposition of these two states,

$$i\hbar\left(\alpha\mathbf{K}_\chi\left|\chi\right\rangle + \beta\mathbf{K}_\varphi\left|\varphi\right\rangle\right) = E\left(\alpha\left|\chi\right\rangle + \beta\left|\varphi\right\rangle\right) = E\left|\Psi\right\rangle , \tag{11}$$

we thus need an "invariant notion of '$\frac{\partial}{\partial t}$' " [24] to talk about what this means in the context of Killing vectors replacing the time-translation operator. We can't identify $\mathbf{K}_\chi$ with $\mathbf{K}_\varphi$ since these are different space-times, and don't just occupy different parts of the same one. Using a 'Newton/Cartan gravitational scheme' (see [7] for an outline) can lead us to an 'absolute time' τ but this on its own does not actually tell us what $\frac{\partial}{\partial \tau}$ is.

[7] then suggests we measure the error that's involved, if we do try to identify the two Killing vectors with each other. If we consider the forces $\mathbf{F}_\chi$ and $\mathbf{F}_\varphi$ at each location, and the corresponding potentials Φ_χ and Φ_φ, then it is proposed that the 'uncertainty' at a time t is given by

$$\begin{aligned}\Delta &= \int (\mathbf{F}_\chi - \mathbf{F}_\varphi)^2 d^3x \\ &= \int \left[\nabla\left(\Phi_\chi - \Phi_\varphi\right)\right]^2 d^3x. \end{aligned} \tag{12}$$

If we use Poisson's formula $\nabla^2\Phi = -4\pi G\rho$, and solve for Φ using a Green's function, then we can write this as

$$\begin{aligned}\Delta &= 4\pi G \int (\Phi_\chi - \Phi_\varphi)(\rho_\chi - \rho_\varphi) d^3x \\ &= -4\pi G \int\int \frac{(\rho_\chi(x) - \rho_\varphi(x))(\rho_\chi(y) - \rho_\varphi(y))}{|x-y|}\, d^3x\, d^3y \end{aligned} \tag{13}$$

which is apparently the "Gravitational self-energy of the *difference* between the mass distributions of each of the two lump locations," E_Δ. By noting that it is time uncertainty that we are interested in, [7] concludes by saying that the superposed state would not actually be stationary, and have a lifetime of the order

$$T_G \simeq \hbar/E_\Delta. \tag{14}$$

by comparison with Heisenberg's uncertainty principle, $\Delta E \Delta t \geq \frac{1}{2}\hbar$. It is also possible that the gravitational interaction energy

$$\tilde{E}_\Delta = G \int\int \frac{\rho_\chi(x)\rho_\varphi(y)}{|x-y|}\, d^3x\, d^3y \tag{15}$$

gives a better measure for the energy uncertainty involved. It will be significantly different from the self-energy in certain situations, although which one we take as a measure of energy uncertainty hasn't yet been resolved (the self-potential energy is usually considered instead).

It should be pointed out that these proposals are taken to be valid in the 'Newtonian' limit of gravity, rather than the 'full-blown' picture provided by general relativity, and that it is freely admitted that there are plenty of problems with this "incompletely motivated" theory [24], such as the above.

There is also the question of what we take to be the eigenstates that don't collapse any further (the 'basic states') since any state can be written as a superposition of others; for example,

$$|\chi\rangle = |\chi\rangle - |\varphi\rangle + |\varphi\rangle = |\varphi\rangle + (|\chi\rangle - |\varphi\rangle), \tag{16}$$

and which are actually involved in such a superposition (any particle can be in a practically infinite number of position eigenstates, and so would have an infinite energy uncertainty!). [8] further proposes that such states are the basic stationary states of the Schrödinger equation with an extra gravitational Newtonian self-potential term included – the Schrödinger-Newton equations – which we shall investigate later in this essay.

1.5 Experimental setups

Actual experimental setups to look for gravitationally induced collapse are proposed in [8, 24], and furthered in [18, 25]. The basic premise is to entangle a small lump of mass with a photon and see if a superposition of lump locations will collapse their wavefunctions.

For example, we could use a laser to fire a photon through another half-silvered mirror or other beam splitter. The transmitted path should have the photon move a small mass (e.g. a tiny mirror) by some method, and then be held somewhere whilst having its state preserved – for example, in a specially build cavity [15], or reflected off a far-away mirror (though using a mirror at the end of the paths would itself . The reflected path should similarly be directed straight into a cavity.

Both states should then be held inside for the amount of time it should take for the mass to spontaneously collapse into one of its position eigenstates, according to our scheme, before being released to travel back along the same paths they took. Meanwhile, the entangled mass is to have some (automatic) restoring force applied to it so that the transmitted part of the photon will reflect off it and return to the mirror, reaching it at the exactly same time as the reflected path.

At this point, if the state has collapsed, part of the transmitted photon will be reflected by the mirror into a detector; but if not, both parts of the photon will simply recombine at the mirror and travel back to the laser, and we will not see anything. Of course, we will need to ensure that the setup does not allow the system to collapse other than through the way we want. One area we should be wary of is the apparatus used to restore the mass to its original location – perhaps we could use another laser, connected to a charged capacitor, to push it back into place with a quick burst of light if triggered, and another is to not let the environment interact excessively with the setup. Penrose has suggested performing this experiment in space [24] with x-ray photons in order to try to avoid such interference, though doing such a thing is expensive and technically very difficult.

Christian has recently suggested that we could study neutrino oscillations to check for gravitational collapse [25], since they only interact via the weak and gravitational forces and so shouldn't be subject to 'environmental' decoherence in the vacuum. It is currently believed that neutrinos flip between their three different 'flavours' – electron, muon and tau – and that they have mass (with tau neutrinos being the most massive), which is needed to explain why we detect fewer neutrinos from the sun than the basic theory predicts we should. The proposal in [25] is that because the different flavours have different masses and will thus travel at different velocities, there is a "gravity-induced ill-definedness" that grows with the distance they have to travel, and from this we should be able to calculate the flux ratios for the different flavours after they have travelled a certain distance. By comparing this with the observed flux ratios for highly-energetic neutrinos from very distant sources – the suggestion is to look at sources around 10^9 light years away, roughly the distance to some quasars – then "the Diósi-Penrose scheme for state reduction can be either ruled out or verified." Of course, neutrino detection is a tricky business, and only preliminary observations have so far been made of such numbers.

1.6 The Schrödinger-Newton equations

The time-dependent Schrödinger equation is

$$H(\mathbf{x},t)\left|\psi(\mathbf{x},t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi(\mathbf{x},t)\right\rangle . \tag{17}$$

We propose that (17) is modified by the addition of an extra gravitational term (as part of H) that corresponds to the gravitational self-energy of the particle, or system, under consideration. That is, we add a term $E_{grav} = m\Phi$ where Φ is the 'self-potential' given by solving Poisson's equation with a Green's function (as usual with gravitational potentials) and mass density ρ:

$$\Phi(\mathbf{x}, t) = -G \int \frac{\rho(\mathbf{y}, t)}{|\mathbf{x} - \mathbf{y}|} \, d^3\mathbf{y} \tag{18}$$

Given that we normally regard $\rho = |\Psi|^2$ as the probability density of the particle(s) in question,[5] it makes sense that $m|\Psi|^2$ represents the mass density – we assume that the mass of the particle is not stuck in one place while the particle itself is 'spread' over a volume! So we propose to rewrite this term as

$$\Phi(\mathbf{x}, t) = -G \int \frac{m|\Psi(\mathbf{y}, t)|^2}{|\mathbf{x} - \mathbf{y}|} \, d^3\mathbf{y} \tag{19}$$

And thus we have the time-dependent SN equations, with some external potential V:

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi + m\Phi\Psi && (20) \\ \nabla^2\Phi &= 4\pi G m|\Psi|^2 && (21) \end{aligned}$$

with the time-independent SN equations the same, but where (20) now reads

$$E\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi + m\Phi\Psi \tag{22}$$

(see the stationary states section later on for proof this can be done).

[5] We can use a method similar to that for the standard Schrödinger equation to easily show that ρ is conserved by the SN equations if it is real-valued, which, given its integral form and physical considerations, we expect it to be.

2 Mathematical results

2.1 Non-dimensionalisation

We can non-dimensionalise the SN equations for a free particle; redefine the fields and length/time scales:

$$\mathbf{x} = L\tilde{\mathbf{x}}, \quad t = T\tilde{t}, \quad \Phi = \alpha\phi, \quad \Psi = \beta\psi, \tag{23}$$

and insert into our equations:

$$\begin{aligned} -\frac{\hbar^2\beta}{2mL^2}\tilde{\nabla}^2\psi + m\alpha\beta\psi\phi &= \frac{i\hbar\beta}{T}\frac{\partial\psi}{\partial\tilde{t}}, \\ \alpha\tilde{\nabla}^2\phi &= 4\pi GmL^2\beta^2|\psi|^2, \\ \int\beta^2|\psi|^2L^3d^3\tilde{x} &= 1. \end{aligned} \tag{24}$$

Balancing all the terms above gives

$$\frac{L^2}{T} = \frac{\hbar}{2m}, \quad \alpha T = \frac{\hbar}{m}, \quad \frac{L^2\beta^2}{\alpha} = \frac{1}{4\pi Gm}, \quad \beta^2 = \frac{1}{L^3}, \tag{25}$$

and so setting

$$\begin{aligned} L = \frac{\hbar^2}{8\pi Gm^3}\,, &\quad T = \frac{\hbar^3}{32\pi^2G^2m^5}\,, \\ \alpha = \frac{32\pi^2G^2m^4}{\hbar^2}, &\quad \beta = \frac{16}{\hbar^3}\sqrt{2\pi^3G^3m^9}, \end{aligned} \tag{26}$$

gives us the non-dimensionalised SN equations

$$-\tilde{\nabla}^2\psi + \phi\,\psi = i\frac{\partial\psi}{\partial\tilde{t}}, \tag{27}$$

$$\tilde{\nabla}^2\phi = |\psi|^2, \tag{28}$$

with normalisation $\int|\psi|^2 = 1$.

We shall also non-dimensionalise the time-independent SN equations; as before, let

$$\mathbf{x} = L\tilde{\mathbf{x}}, \quad E = \mathcal{E}\tilde{E}, \quad \Phi = \alpha\phi, \quad \Psi = \beta\psi. \tag{29}$$

OBJECT	MASS (kg)	L	T	α	β
Electron neutrino	10^{-35}	10^{45}	10^{114}	10^{-90}	10^{-69}
Hydrogen atom	10^{-27}	10^{21}	10^{73}	10^{-59}	10^{-32}
5 femtograms	5×10^{-18}	10^{-10}	10^{26}	10^{-21}	10^{11}
E. Coli	10^{-16}	10^{-14}	10^{19}	10^{-11}	10^{20}
Human cell	10^{-12}	10^{-25}	0.5	127	10^{35}
Planck mass	10^{-8}	10^{-38}	10^{-22}	10^{19}	10^{54}
Schrödinger's cat	5	10^{-63}	10^{-64}	10^{53}	10^{92}

Table 1: Values of non-dimensional parameters for various objects

Set the scalings to be the same as with the time-dependent case, but with

$$\mathcal{E} = \frac{32\pi^2 G^2 m^5}{\hbar^2} \tag{30}$$

replacing the time term, to get

$$\begin{aligned} -\tilde{\nabla}^2\psi + \phi\,\psi &= E\psi, & (31)\\ \tilde{\nabla}^2\phi &= |\psi|^2, & (32) \end{aligned}$$

again with the same normalisation for ψ. In SI units our scalings for the time-dependent equations are

$$\begin{aligned} L = 6.57 \times 10^{-60}\, m^{-3}, &\qquad T = 5.23 \times 10^{-61}\, m^{-5},\\ \alpha = 1.27 \times 10^{50}\, m^{4}, &\qquad \beta = 5.93 \times 10^{88}\, m^{\frac{9}{2}}, \end{aligned} \tag{33}$$

where we have omitted the units of measurement (m is the mass of the object under consideration, and doesn't stand for meters). See table 1 for scalings of various objects.

Note the scalings for T: even 10^{19} seconds is 100 times longer than the age of the Universe, and so we should expect tiny objects to evolve incredibly slowly with time. But why are the length scales so large for the smallest particles? Probably because of the Poisson-like part of the SN equations, since their gravitational self-energy is so small and we are trying to balance this with other terms in the equations: we should perhaps be using the standard Schrödinger equation for such instances, and reserve the SN equations for objects larger than an atom.

OBJECT	MASS (kg)	SIZE (m)	T	α	β
Proton	10^{-27}	10^{-15}	10^{14}	10^{-22}	10^{22}
Hydrogen atom	10^{-27}	10^{-11}	10^{18}	10^{-26}	10^{16}
Microtubule	10^{-22}	10^{-8}	10^{11}	10^{-24}	10^{12}
E. Coli	10^{-16}	10^{-6}	40	10^{-11}	10^{9}
Human cell	10^{-12}	10^{-5}	10^{-6}	10^{-17}	10^{7}
Ant	10^{-6}	10^{-2}	10^{-18}	10^{-13}	1000
Schrödinger's cat	5	0.4	10^{-27}	10^{-10}	4
'Normal' black hole	10^{30}	6000	10^{-82}	10^{17}	10^{-6}

Table 2: Taking given length scales into account; some values given by [27]. Mass of a typical microtubule is given by Penrose [5] to be about 10^{-14} in 'absolute units' (where the Planck mass is set to be one of these units) and the length scale given is the exterior diameter ($\simeq$ 25nm).

But suppose we consider the length scales that are relevant to the objects we are studying and see if terms turn out small, rather than trying to balance all the terms together; if we start by balancing the normalisation and Poisson terms together,

$$\beta^2 = \frac{1}{L^3}, \qquad \alpha = \frac{4\pi G m}{L}, \tag{34}$$

and inserting into a external potential-less (20),

$$\phi\psi - \frac{\hbar^2}{8\pi G m^3 L}\nabla^2\psi = \frac{i\hbar L}{4\pi G m^2 T}\frac{\partial\psi}{\partial t}. \tag{35}$$

If we consider an E. Coli bacteria, which has a length of about 3×10^{-6}m, and insert the value for our mass, we get

$$\phi\psi - (2 \times 10^{-6})\nabla^2\psi = \frac{40i}{T}\frac{\partial\psi}{\partial t}. \tag{36}$$

So if we set $T = 40$ seconds, we 'balance' the first and last terms (what we do with the term in i is a bit tricky, since we can't rescale β to be complex or negative as it's linked to L by normalisation!). See table 2 for more scalings taking this into account. If instead we consider the time-independent problem and balance the energy term with $\phi\psi$, we'd have an energy scaling

$$\mathcal{E} = \frac{4\pi G m^2}{L} \tag{37}$$

where we note that

$$\frac{\hbar}{\mathcal{E}} = T = \frac{\hbar L}{4\pi G m^2} \tag{38}$$

which looks something like Penrose's proposed time scale in which gravity causes wavefunction collapse. However in [24] he calculates that a mirror of mass 5×10^{-12}kg and 10^{-5}m in diameter should collapse in about 0.1 seconds: the above rescaling would suggest $T = 5 \times 10^{-8}$ seconds, so unfortunately this scaling doesn't correspond to collapse times (if we go back to our previous scalings where just the mass was set, we get a better figure of 10^{-4} seconds). Fixing the length, mass and time scales in the non-dimensionalised equations to be these values doesn't offer any insight either into how we should scale α, β or what terms should be balanced in the run up to a collapse, if indeed we can link these equations with collapse.

In any case, we have not really justified balancing the terms that we have – for objects that don't have their wavefunction 'smeared out' but are instead fairly well-defined, $\nabla^2\psi$ should actually be quite large at their boundaries (it's reasonable to assume that this is the case for a well defined macroscopic object) – nor have we really said what the time-scale corresponds to physically. But in the time-independent equations the $\phi\psi$-∂_t scalings mean that we can effectively ignore the $\nabla^2\psi$ term because it is so small, so that the scaled energy eigenvalue corresponds to the gravitational self-potential – that is, for microscopic objects we have

$$\phi\psi \simeq E\psi. \tag{39}$$

2.2 The ground state energy

In a paper [16] Tod calculated an upper bound on the ground state energy of the time-independent SN equations, and a brief outline of his work is given below. The ground state energy we are interested in is the lowest, non-zero value of E given by the free particle, time-independent SN equations (31), and so we want to minimise $H\psi$. We can use calculus of variations for this task, using the normalisation of ψ as an integral constraint.

Consider the Euler-Lagrange action

$$I[\gamma] \;=\; \int L(\psi, \phi, \nabla\psi) d^3x$$

$$= \int \left[\frac{1}{2}\nabla\psi \cdot \nabla\overline{\psi} + \frac{1}{4}\phi|\psi|^2 + \lambda|\psi|^2\right] d^3x \tag{40}$$

which we will want to use to minimise the energy eigenvalue E, and where we note ϕ is given by (32) and λ is a Lagrange multiplier. Now consider the variation of this action and ignore $O(\delta^2)$:

$$\begin{aligned}
\delta I &= I[\gamma + \delta\gamma] - I[\gamma] \\
&= \int_V \left[L(\psi + \delta\psi, \phi + \delta\phi, \nabla(\psi + \delta\psi)) - L(\psi, \phi, \nabla\psi)\right] d^3x \\
&= \int_V \left[\frac{1}{2}\left(\nabla\delta\psi \cdot \nabla\overline{\psi} + \nabla\psi \cdot \nabla\delta\overline{\psi}\right)\right. \\
&\quad + \frac{1}{4}\left(\delta\phi\, \nabla^2\phi + \phi\left[\psi\, \delta\overline{\psi} + \overline{\psi}\nabla\delta\psi\right]\right) \\
&\quad \left. + \lambda\left(\delta\psi\, \overline{\psi} + \psi\, \delta\overline{\psi}\right)\right] d^3x \\
&= \int_V \frac{1}{2}\left[\nabla \cdot \left(\delta\psi\, \nabla\overline{\psi} + \delta\overline{\psi}\, \nabla\psi\right) - \left(\delta\psi\, \nabla^2\overline{\psi} + \delta\overline{\psi}\, \nabla^2\psi\right)\right. \\
&\quad + \frac{1}{2}\nabla \cdot (\delta\phi\nabla\phi - \phi\nabla\delta\phi) + \phi\left(\delta\psi\, \overline{\psi} + \psi\, \delta\overline{\psi}\right) \\
&\quad \left. + 2\lambda\left(\delta\psi\, \overline{\psi} + \psi\, \delta\overline{\psi}\right)\right] d^3x \\
&= \int_{\delta V} \frac{1}{2}\left[\delta\psi\, \nabla\overline{\psi} + \delta\overline{\psi}\, \nabla\psi + \frac{1}{2}\left(\delta\phi\nabla\phi - \phi\nabla\delta\phi\right)\right] \cdot d\underline{S} \\
&\quad + \int_V \delta\psi\left(\frac{1}{2}\phi\overline{\psi} - \nabla^2\overline{\psi} + \lambda\overline{\psi}\right) d^3x \\
&\quad + \int_V \delta\overline{\psi}\left(\frac{1}{2}\phi\psi - \nabla^2\psi + \lambda\psi\right) d^3x
\end{aligned} \tag{41}$$

and hence, with $\delta\psi = \delta\phi = ... = 0$ on the boundary δV this gives the action for (27). Using Sobolev and Hölder inequalities, one can then prove that I is bounded below by $-\frac{1}{54\pi^2}$ (remaining in non-dimensionalised units).

However, if we multiply (31) by $\overline{\psi}$ and integrate over the space, using the divergence theorem and the normalisation condition again, we note that I is not equal to the energy:

$$E = \int \left(-\overline{\psi}\nabla^2\psi + \phi\psi\overline{\psi}\right) d^3x$$

$$= \int \left(|\nabla\psi|^2 - \nabla\left[\overline{\psi}\nabla\psi\right] + \phi\psi\overline{\psi}\right) d^3x$$
$$= \int \left(|\nabla\psi|^2 + \phi|\psi|^2\right) d^3x \tag{42}$$

It can be shown (e.g. by defining an appropriate tensor which satisfies the conservation equation) that

$$T = \int |\nabla\psi|^2 = -\frac{1}{3}E, \quad V = \int \phi|\psi|^2 = \frac{4}{3}E \tag{43}$$

(These integrals appear to correlate to the kinetic and potential energies, and [17] points out that (32) implies the total energy $\mathcal{E} = T + V$ is not conserved by the SN equations.) Using the energy scaling (30), we have that

$$E = 6I \geq -\frac{1}{9\pi^4} = -\frac{32G^2m^5}{9\pi^2\hbar^2}. \tag{44}$$

2.3 Stationary States

With a Φ that's independent of time we can consider stationary states in the same way we do with the ordinary Schrödinger equation. However, given its form (19) this isn't the case and we will end up with a time-dependent Hamiltonian, unless Ψ is also independent or time dependence is lost in the integral. We shall assume that the latter is true, and consider separable solutions of the form $\psi(\mathbf{x}, t) = \Psi(\mathbf{x})\mathcal{T}(t)$. Inserting into (20):

$$i\hbar\Psi\frac{d\mathcal{T}}{dt} = H\Psi\mathcal{T} = \left(-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi + m\Phi\Psi\right)\mathcal{T}. \tag{45}$$

Divide by $\Psi\mathcal{T}$, and by separation of variables both sides must be constant:

$$i\hbar\frac{1}{\mathcal{T}}\frac{d\mathcal{T}}{dt} = \frac{H\Psi}{\Psi} = \text{ constant } E, \tag{46}$$

which we can use to calculate $\mathcal{T}(t)$, and so

$$\psi(\mathbf{x}, t) = \Psi(\mathbf{x})e^{-iEt/\hbar} \tag{47}$$

where E is an energy eigenvalue satisfying $H\Psi = E\Psi$. We therefore don't have a problem with Φ being time dependent since this phase factor $e^{-iEt/\hbar}$

has constant magnitude, i.e. that $|\psi(\mathbf{x},t)|^2 \equiv |\Psi(\mathbf{x})|^2$.

If the potential $V(\mathbf{x})$ satisfies $\nabla^2 V = 0$ (e.g. linear or no potential) then we can write the stationary SN equations as

$$\nabla^2\Psi = -\frac{2m}{\hbar^2}\,\Omega\Psi \tag{48}$$

$$\nabla^2\Omega = -4\pi G m\,|\Psi|^2 \tag{49}$$

where

$$\Omega = \frac{(E-V)}{m} - \Phi. \tag{50}$$

To rescale, set

$$\Psi = \sqrt{\frac{\hbar^2}{8\pi G m^3}}\,S, \qquad \Omega = \frac{\hbar^2}{2m}\,T \tag{51}$$

so that

$$\begin{aligned} \nabla^2 S &= -ST \\ \nabla^2 T &= -S^2 \end{aligned} \tag{52}$$

with S real (without loss of generality). This result, which is also mentioned in [10], is quite a useful form and is one that we can study numerically analytically much more easily.

2.4 Spherically-symmetric, stationary solutions

Numerical studies of both the time-dependent and -independent equations have been carried out in [17, 19, 20] and to a lesser extent in [10]. Whilst focusing mainly on the spherically-symmetric form, [20] also contains an analysis of an axially symmetric state, and a two-dimensional state. The below is a brief overview of the work carried out in these papers.

We start with equations (52) and consider S, T as functions of radius r only;

$$\begin{aligned} \frac{1}{r}\,\frac{d^2(rS)}{dr^2} &= -ST, \\ \frac{1}{r}\,\frac{d^2(rT)}{dr^2} &= -S^2, \end{aligned} \tag{53}$$

which have 'obvious' solutions that we are not interested in: ($S = 0, T =$ constant) (trivial) and ($S = \pm\frac{2}{r^2}, T = -\frac{2}{r^2}$) (blows up at $r = 0$). We'd like to search for solutions that are finite at 0, and go to 0 as $r \to \infty$; this problem is proved to be "well-posed" in [12].

An integral form of the equations can be found by integrating (53) twice and using the replacement lemma:[6]

$$\begin{aligned}
\int_0^r [xS(x)]'' dx &= -\int_0^r xS(x)T(x)dx \\
\Rightarrow [S(x) + xS'(x)]_0^r &= -\int_0^r xS(x)T(x)dx \\
\Rightarrow \int_0^r [xS(x)]' dx &= rS(0) - \int_0^r \int_0^y xS(x)T(x)\, dx\, dy \\
\Rightarrow S(r) &= S(0) - \int_0^r x\left(1 - \frac{x}{r}\right) S(x)T(x)dx \\
T(r) &= T(0) - \int_0^r x\left(1 - \frac{x}{r}\right) S(x)^2 dx
\end{aligned} \tag{54}$$

Note that differentiating S and T gives

$$\begin{aligned}
S'(r) &= -\int_0^r \frac{x^2}{r^2} S(x)T(x)dx \\
T'(r) &= -\int_0^r \frac{x^2}{r^2} S(x)^2 dx < 0
\end{aligned} \tag{55}$$

and hence $T(r) = \frac{2m}{\hbar^2}\left[\frac{(E-V)}{m} - \Phi\right]$ is monotonically decreasing with r (something we could also see by realising that $x(1 - \frac{x}{r}) \geq 0$ for $0 \leq x \leq r$). Thus, if $T(0) \leq 0$, then $T \leq 0$ too; this implies that S is increasing if $S(0) \geq 0$, and it is possible to prove that in fact $S \to \infty$ while $T \to -\infty$ if this is the case. And if $T(0) \geq 0$, then [10] also claims that rS needs to have exponentially decaying terms if it is not to blow up (although it this paper also shows (numerically) that S does not go to $\pm\infty$ if we start with a specific value of $S(0) \in [1.08, 1.09]$).

[12] contains a proof that if $S \to \infty$ and $T \to -\infty$ then they do so in finite

[6] This is the identity $\int_0^r \int_0^y f(x)\, dx\, dy = \int_0^r (r - y) f(y)\, dy$ with smooth f, etc.

r: by considering the quantity

$$Q(r) = r(S+T), \quad X = rS, \tag{56}$$

and a "modified energy"

$$\mathcal{F}(r) = \frac{(Q')^2}{S} + Q^2, \tag{57}$$

which decreases with r (since $\mathcal{F}'(r) < 0$), they show via (53) and several inequalities using the above identities that

$$X'' > \frac{X^2}{2r} > 0 \tag{58}$$

for sufficiently large r. Integrating this and using more inequalities gives

$$S > \frac{6}{(r-r_0)^2} \tag{59}$$

which means S to blow up at $r = r_0 =$ constant; the argument also shows the same for if $S \to -\infty$.

2.5 Stability

In order to consider the stability of stationary states of the spherically-symmetric time-dependent SN equations, [19] tries a perturbation using a small parameter ϵ:

$$\begin{aligned} \Psi &= \psi_1 + \epsilon\psi_1 + O(\epsilon^2) \\ \Phi &= \phi_0 + \epsilon\phi_1 + O(\epsilon^2). \end{aligned} \tag{60}$$

Substitute into (27,28) and equate powers of ϵ:

$$\begin{aligned} i\psi_{0,t} &= -\nabla^2\psi_0 + \psi_0\phi_0 \\ |\psi_0|^2 &= \nabla^2\phi_0 \end{aligned} \tag{61}$$

$$\begin{aligned} i\psi_{1,t} &= -\nabla^2\psi_1 + \psi_0\phi_1 + \psi_1\phi_0 \\ \psi_0\overline{\psi}_1 + \overline{\psi}_0\psi_1 &= \nabla^2\phi_1 \end{aligned} \tag{62}$$

(where the subscript $_{,t}$ is a comma derivative representing $\frac{\partial}{\partial t}$.) Since we are perturbing stationary states, set

$$\begin{aligned} \psi_0 &= R_0(r)e^{-iEt}, \\ \phi_0 &= E - T_0(r), \end{aligned} \tag{63}$$

with R_0, T_0 real valued (without loss of generality) and insert into (62), using the spherical polar form of ∇^2:

$$\begin{aligned} i\psi_{1,t} &= -\frac{1}{r}(r\psi_1)_{,rr} + \phi_1 R_0(r)e^{-iEt} + (E - T_0(r))\psi_1 \\ \frac{1}{r}(r\phi_1)_{,rr} &= R_0(r)e^{-iEt}\overline{\psi}_1 + R_0(r)e^{iEt}\psi_1. \end{aligned} \tag{64}$$

Solutions of the form

$$\begin{aligned} \psi_1(r,t) &= \frac{1}{r}\,R_1(r,t)e^{-iEt}, \\ \phi_1(r,t) &= \frac{1}{r}\,\phi_2(r,t) \end{aligned} \tag{65}$$

with

$$\begin{aligned} R_1 &= (A(r) + B(r))e^{\lambda t} + (\overline{A}(r) - \overline{B}(r))e^{\overline{\lambda} t}, \\ \phi_2 &= W(r)e^{\lambda t} + \overline{W}(r)e^{\overline{\lambda} t} \end{aligned} \tag{66}$$

inserted into (64) has

$$\begin{gathered} i\lambda(A+B)e^{\lambda t} + i\overline{\lambda}(\overline{A} - \overline{B})e^{\overline{\lambda} t} + (A_{rr} + B_{rr})e^{\lambda t} \\ +(\overline{A}_{rr} - \overline{B}_{rr})e^{\overline{\lambda} t} = R_0 W e^{\lambda t} + R_0 \overline{W} e^{\overline{\lambda} t} - U_0(A+B)e^{\lambda t} \\ -U_0(\overline{A} - \overline{B})e^{\overline{\lambda} t}, \end{gathered}$$

$$\begin{gathered} R_0(\overline{A} + \overline{B})e^{\overline{\lambda} t} + R_0(A - B)e^{\lambda t} + \overline{R}_0(A+B)e^{\lambda t} \\ +\overline{R}_0(\overline{A} - \overline{B})e^{\overline{\lambda} t} = W_{rr}e^{\lambda t} + \overline{W}_{rr}e^{\overline{\lambda} t}. \end{gathered} \tag{67}$$

Equating co-efficients of $e^{\lambda t}$ and $e^{\overline{\lambda} t}$ in these gives the perturbation equations

$$\begin{aligned} W_{rr} &= 2R_0 A, \\ A_{rr} + U_0 A - R_0 W &= -i\lambda B, \end{aligned}$$

$$B_{rr} + U_0 B = -i\lambda A. \quad (68)$$

[19] also notes that reversing the sign on B also reverses the sign of λ, giving eigenvalues to this system to be $(\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda})$.

To solve this, they use a spectral method to turn this into a problem involving matrices, which makes it well suited for a numerical analysis with, for example, Matlab. From this, they determine that the eigenvalues of perturbations around the ground state are purely imaginary and thus this state is (linearly) stable, and that the n-th excited state above this has n unstable modes (with eigenvalues that are not purely imaginary). These higher states decay from an arbitrary initial configuration into the ground state, but have a normalisation lower than 1 (see, for example, [17] figure 6.21). This certainly has consequences for the SN equations giving us basis states.

3 Lie Point Symmetries

3.1 Introduction and Derivation

We start with a system of 2nd-order partial differential equations that need investigating:

$$H_A(x^i, u^\beta, u^\beta_{,n}, u^\beta_{,nm}) = H_A(y^a) = 0. \quad (69)$$

A point symmetry of a (partial) differential equation is a transformation $\tilde{y}^a = \tilde{y}^a(y^b)$ with $H_A(\tilde{y}^a) = 0$; that is, it is a transformation that leaves a family of solutions invariant.

If we can calculate the symmetries of a PDE, then it is possible that we can use them to construct solutions, or at least see how the system behaves. One such route we can take is to find the Lie point symmetries of the system, which we shall explain and define below. Many books have been written on this topic (for example, Stephani [4], from which the techniques in this paper are understood) so only a brief introduction to point symmetries and the application to 2nd order PDEs will described here, and applied in detail later.

To proceed, we consider an infinitesimal point transformation (there are other types we could consider, such as contact transformations) of the independent and dependent variables x^n and u^α to new variables:

$$\begin{aligned} \tilde{x}^n &= x^n + \varepsilon \mathbf{X} x^i + O(\varepsilon^2) = x^n + \varepsilon \xi^n(x^i, u^\beta) + ... \\ \tilde{u}^\alpha &= u^\alpha + \varepsilon \mathbf{X} u^\beta + O(\varepsilon^2) = u^\alpha + \varepsilon \eta^\alpha(x^i, u^\beta) + ... \end{aligned} \quad (70)$$

where

$$\xi^n := \left.\frac{\partial \tilde{x}^n}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad \eta^\alpha := \left.\frac{\partial \tilde{u}^\alpha}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad (71)$$

and the 'infinitesimal generator' $\mathbf{X}$ is given by

$$\mathbf{X} = \xi^n \frac{\partial}{\partial x^n} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_n \frac{\partial}{\partial u^\alpha_{,n}} + \eta^\alpha_{nm} \frac{\partial}{\partial u^\alpha_{,nm}}, \quad (72)$$

with similar definitions

$$\eta^\alpha_n = \left.\frac{\partial \tilde{u}_{,n}}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \eta^\alpha_{nm} = \left.\frac{\partial \tilde{u}_{,nm}}{\partial \varepsilon}\right|_{\varepsilon=0}. \quad (73)$$

A few words on the notation: x^i represents the range of M independent variables $\{x^1, .., x^M\}$, u^β represents the range of N dependent variables $\{u^1, .., u^N\} \equiv \{u, v, w...\}$ (to which Greek letter indices are reserved), and x^n and u^α are specific variables. Comma derivatives and Einstein notation are also employed (where repeated 'upstairs' and 'downstairs' indices are summed over).

To calculate η^α_n and η^α_{nm} we start by differentiating (70):

$$\begin{aligned} d\tilde{u}^\alpha &= du^\alpha + \varepsilon d\eta^\alpha + ... \\ &= \left[\frac{\partial u^\alpha}{\partial x^i} + \varepsilon\left(\frac{\partial \eta^\alpha}{\partial x^i} + \frac{\partial \eta^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^i}\right)\right] dx^i + ... \end{aligned} \tag{74}$$

$$\begin{aligned} d\tilde{x}^n &= dx^n + \varepsilon d\xi^n + ... \\ &= \left[\delta^n_m + \varepsilon\left(\frac{\partial \xi^n}{\partial x^i} + \frac{\partial \xi^n}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^m}\right)\right] dx^i + ... \end{aligned} \tag{75}$$

If we divide both these terms to find $\frac{\partial \tilde{u}^\alpha}{\partial \tilde{x}^n}$ and compare with $\frac{\partial \tilde{u}^\alpha}{\partial \tilde{x}^n}$ calculated directly from (70), and similarly to second order, it can be shown that

$$\eta^\alpha_n = \eta^\alpha_{,n} + \eta^\alpha_{,\beta}u^\beta_{,n} - \xi^i_{,n}u^\alpha_{,i} - \xi^i_{,\beta}u^\beta_{,n}u^\alpha_{,i}, \tag{76}$$

$$\begin{aligned} \eta^\alpha_{nm} = {} & \eta^\alpha_{,nm} + \eta^\alpha_{,n\beta}u^\beta_{,m} + \eta^\alpha_{,m\beta}u^\beta_{,n} - \xi^i_{,nm}u^\alpha_{,i} \\ & + \eta^\alpha_{,\beta\gamma}u^\beta_{,n}u^\gamma_{,m} - u^\alpha_{,k}\left(\xi^k_{,n\beta}u^\beta_{,m} + \xi^k_{,m\beta}u^\beta_{,n}\right) \\ & - \xi^k_{,\beta\gamma}u^\beta_{,n}u^\gamma_{,m}u^\alpha_{,k} + \eta^\alpha_{,\beta}u^\beta_{,nm} - \xi^i_{,n}u^\alpha_{,mi} - \xi^i_{,m}u^\alpha_{,ni} \\ & - \xi^k_{,\beta}\left(u^\alpha_{,k}u^\beta_{,nm} + u^\beta_{,n}u^\alpha_{,mk} + u^\alpha_{,nk}u^\beta_{,m}\right). \end{aligned} \tag{77}$$

Finally, we can rewrite (72) and use (71) to show that

$$\mathbf{X} = (\mathbf{X}y^a)\frac{\partial}{\partial y^a} = \left.\frac{\partial \tilde{y}^a}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial y^a}, \tag{78}$$

which gives

$$\begin{aligned} \mathbf{X}H_A &= \frac{\partial H_A}{\partial y^a}\left.\frac{\partial \tilde{y}^a}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= \frac{\partial H_A}{\partial \tilde{y}^a}\left.\frac{\partial \tilde{y}_a}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &= \left.\frac{\partial H_A}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}H_A(\tilde{y}^a) \end{aligned}$$

$$= \quad 0. \tag{79}$$

This result, that $\mathbf{X}H_A = 0$ for the symmetry generator $\mathbf{X}$, is an important one that will be the first step in calculating symmetries.

Although this whole process looks nasty, we can simplify the process by first calculating the terms in (76,77) for terms that don't appear in $H_A = 0$ and eliminating; then considering only the second order derivatives; and then finally by trying the whole thing out; all the while using the original condition $H_A = 0$ and, as Stephani puts it, "... the symmetry condition has to be satisfied *identically* in all remaining variables $(x^i, u^\alpha, u^\alpha_{,i}, ...)$ since it has to be true for every solution, which means that arbitrary values can be assigned to all these variables" (but being extra careful as to when we apply this).

This paper will find the symmetries of the heat conduction equation as a 'warm-up' (this particular example is mentioned in Stephani [4] chapter 16, but only the solution is given), and follow by finding the symmetries of the SN equations, which have not yet been published fully (though they are briefly mentioned in [17]).

3.2 Using the calculated symmetries

After we have calculated ξ and η, we then need to work out what sort of symmetries our system admits, and see if we can use them to solve the original equation. To do the former, we can simply use the framework involving these infinitesimal point transformations that we have already set up (called the 'method of finite symmetry transformations'). If

$$\mathbf{X}H = \xi^n \frac{\partial H}{\partial x^n} + \eta^\alpha \frac{\partial H}{\partial u^\alpha} + \eta^\alpha_n \frac{\partial H}{\partial u^\alpha_{,n}} + \eta^\alpha_{nm} \frac{\partial H}{\partial u^\alpha_{,nm}} = 0 \tag{80}$$

only comes out as a first order partial differential equation, we can use the method of characteristics to see how solutions of H are being transformed, and thus generate new ones. We read off what ξ^i, η^α etc. are from the calculated form (80), and use (71) to solve for $\tilde{x}^n$ and $\tilde{u}^\alpha$ by creating appropriate solution surfaces $\tilde{u}^\alpha$ parameterised by ε.

For example, imagine we have some system that ends up looking like

$$\mathbf{X}H = \left(2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}\right) H(x, y, u(x, y)) = 0 \tag{81}$$

Create a solution surface $\tilde{u}(\varepsilon)$ with co-ordinates parameterised by $\tilde{x}(\varepsilon), \tilde{y}(\varepsilon)$; our characteristic curves are given by $(\tilde{x}(\varepsilon), \tilde{y}(\varepsilon), \tilde{u}(\varepsilon))$ and define an initial data curve (x, y, u) on $\varepsilon = 0$. Using Charpit's equations, we see from (81) that

$$\frac{d\tilde{x}}{d\varepsilon} = 2, \quad \frac{d\tilde{y}}{d\varepsilon} = \tilde{x}, \quad \frac{d\tilde{u}}{d\varepsilon} = \tilde{u} \tag{82}$$

which we can solve by integrating and using the initial data:

$$\tilde{x} = x + 2\varepsilon, \quad \tilde{y} = y + \varepsilon(x + \varepsilon), \quad \tilde{u} = ue^{\varepsilon} \tag{83}$$

Thus, $\mathbf{X}$ corresponds to translations for x, y, and a scaling for u. Now suppose that (x, y, u) is some solution to our original equation $H = 0$; because $\mathbf{X}$ is a transformation that generates symmetry we have that $(\tilde{x}, \tilde{y}, \tilde{u})$ is also a solution.

For higher order $\mathbf{X}$, there are other techniques we can employ to discover the type of transformations involved and generate more solutions. But as we shall see, all the generators that this paper deals with are only to first order derivatives; any reader wishing to learn more should read the last few chapters of Stephani's book [4] or other related work.

However, there are two caveats: we need to know a solution to find some more with this method, and even then the new solutions will remain 'close' to the original. But it might be possible to use a 'multiple reduction' process to turn the SN equations – a system of three equations in seven (effectively) independent variables – into an ODE that we might be able to solve, using the symmetries to calculate invariants. Stephani [4] chapters 18 & 19 describe such a technique, which shall be outlined here quickly (without proofs).

We start by looking for the normal form of a symmetry generator

$$\mathbf{X} = \frac{\partial}{\partial \chi} \tag{84}$$

and construct $N+M$ new variables $\chi(x^i, u^\beta)$ and $\varphi^\Omega(x^i, u^\beta)$ that satisfy

$$\mathbf{X}\chi = 1, \qquad \mathbf{X}\varphi^\Omega = 0 \tag{85}$$

for $\Omega = 1, ..., N+M-1$ (where N is the number of independent variables x^i and M the dependent variables u^β). This scheme should reduce $H_A = 0$ into a new system $G_A = 0$ which is independent of χ.

To calculate the φ^Ω, we can use the finite symmetry transformations that correspond to $\mathbf{X}$ (although we may be able to see what the similarity variables are straight away, doing so should give us If possible, one of the equations $\tilde{x}^a = \tilde{x}^a(x^i, u^\beta; \varepsilon)$ should be rearranged so that ε is a function of just one of the transformed independent co-ordinates, and 'original' variables, and we can use this to represent χ:

$$\chi = \varepsilon = \varepsilon(\tilde{x}^a, x^i, u^\beta) \tag{86}$$

If we replace ε in our finite symmetry transformations, we can read off the φ^Ω by making them satisfy

$$\begin{aligned} \tilde{x}^n &= \tilde{x}^n(x^i, u^\beta; \varepsilon(\tilde{x}^a, x^i, u^\beta)) = \tilde{x}^n(\tilde{x}^a, \varphi^\Omega), \\ \tilde{u}^\alpha &= \tilde{u}^\alpha(x^i, u^\beta; \varepsilon(\tilde{x}^a, x^i, u^\beta)) = \tilde{u}^\alpha(\tilde{x}^a, \varphi^\Omega) \end{aligned} \tag{87}$$

We then choose some of the φ^Ω to be new dependent variables w^α, and the rest to be independent variables y^k (the former are usually chosen by considering the equations with u^β in them), and use the chain rule to determine what the original system $H_A = 0$ is in these new co-ordinates.

So, using (81) again as an example, we rearrange (83) so that

$$\varepsilon = \frac{\tilde{x} - x}{2} \quad \Rightarrow \quad \tilde{y} = \frac{\tilde{x}^2 - x^2}{4} + y,$$

$$\tilde{u} = ue^{\frac{1}{2}(\tilde{x} - x)}. \tag{88}$$

From this, we see

$$\varphi^1 = y - \frac{1}{4}x^2, \qquad \varphi^2 = ue^{-\frac{1}{2}x}. \tag{89}$$

Set $\varphi^1 = z$ to be the new independent variable, and $\varphi^2 = w(z)$ to be the new dependent variable, and rearrange (89) and use the chain rule to calculate derivatives:

$$\begin{aligned} u &= w(z)e^{\frac{1}{2}x} \\ \Rightarrow \frac{\partial u}{\partial x} &= \left(\frac{dw}{dz}\frac{dz}{dx} + \frac{1}{2}w(z)\right)e^{\frac{1}{2}x} = \frac{1}{2}\left(-x\frac{dw}{dz} + w(z)\right)e^{\frac{1}{2}x}, \end{aligned} \tag{90}$$

and similarly for other derivatives in whatever the original system $H = 0$ was, to turn it into an ODE. (One should find that the 'original' variables that appear in such derivatives, such as x in our example here, either cancel out or can be easily rewritten in terms of the new co-ordinates and do not appear in $G_A = 0$.)

This method allows us to reduce by one variable at a time, so it should be possible to apply it however many times we need to in order to reduce into an ODE. To do so, we obviously need to know the symmetries of the each new reduced system, but thankfully we do not need to recalculate them every time. Some of the original symmetries may be 'inherited' by the new system; such a $\mathbf{Y}$ must satisfy

$$[\mathbf{X}, \mathbf{Y}] = F(\chi, y^k, w^\alpha)\mathbf{X}. \tag{91}$$

where F is an arbitrary function (possibly $\equiv 0$) and $\mathbf{X}$ is the symmetry we considered beforehand (see Stephani [4] chapter 19). For a further reduction, another inherited Lie point symmetry $\mathbf{Z}$ must commute with both $\mathbf{X}$ and $\mathbf{Y}$. But as there should be other new symmetries for the reduced equations, this would make calculating them worthwhile if we become stuck at this point.

3.3 Example: The heat conduction equation

The rescaled one-dimensional heat conduction equation with $t = x^0$ and $x = x^1$ is

$$H = u_{,11} - u_{,2} = 0. \tag{92}$$

We start by using (72) to give the symmetry condition

$$\mathbf{X}H = \eta_{11} - \eta_2 = 0. \tag{93}$$

We calculate for each of the $\eta^\alpha_{,ij}$ in our symmetry condition the second order terms *not* in (92) using relation (77) – so in this case we need to calculate the terms in η_{11} containing $u_{,12}$ and $u_{,22}$:

$$\eta_{11} = -2\xi^2_{,1}u_{,12} - 2\xi^2_{,u}u_{,1}u_{,12} + ... \tag{94}$$

Considering the symmetry condition only to second order derivatives, and noting that η_2 doesn't contain any second order derivatives, gives that

$$\eta_{11} = u_{,12}\left(\xi^2_{,1} + \xi^2_{,u}u_{,1}\right) + ... = 0, \tag{95}$$

so

$$\xi^2_{,1} = 0, \qquad \xi^2_{,u} = 0, \tag{96}$$

as the second order terms above cannot be eliminated by using $H = 0$.

Next, we compute for each of the $\eta^\alpha_{,ij}$ appearing in $\mathbf{X}H_A = 0$ the remaining $u^\alpha_{,ij}$ terms, and insert into the symmetry condition (again only considered to second order):

$$\eta_{11} = \eta_{,u}u_{,11} - 2\xi^1_{,1}u_{,11} - 3\xi^1_{,u}u_{,1}u_{,11} + ... = 0. \tag{97}$$

We now consider $H_A = 0$ only to second order and apply to the above equation to get more conditions. In this example we have that $u_{,11} = 0$ to second order, so we can't calculate any more useful conditions here (though with other equations we may learn something about η).

Finally, we compute η_{11} and η_2 "fully" and insert into the symmetry conditions $\mathbf{X}H_A = 0$:

$$\begin{aligned}
\eta_{11} - \eta_2 &= \eta_{,11} + 2\eta_{,1u}u_{,1} - \xi^1_{\ ,11}u_{,1} + \eta_{,uu}u^2_{,1} - 2\xi^1_{\ ,1u}u^2_{\ ,1} \\
&\quad - \xi^1_{\ ,uu}u^3_{\ ,1} + \eta_{,u}u_{,11} - 2\xi^1_{\ ,1}u_{,11} - 3\xi^1_{\ ,u}u_{,1}u_{,11} \\
&\quad -\eta_{,2} - \eta_{,u}u_{,2} + \ \xi^1_{\ ,2}u_{,1} + \xi^2_{\ ,2}u_{,2} \\
&= 0. \qquad (98)
\end{aligned}$$

Rearranging;

$$\begin{aligned}
&\eta_{,11} - \eta_{,2} + u_{,1}\left(2\eta_{,1u} - \xi^1_{,11} + \xi^1_{\ ,2}\right) \\
+u^2_{,1}(\eta_{,uu} - 2\xi^1_{\ ,1u}) &- \xi^1_{\ ,uu}u^3_{,1} + u_{,2}\left(-\eta_{,u} + \xi^2_{\ ,2}\right) \\
&+ u_{,11}\left(\eta_{,u} - 2\xi^1_{\ ,1} - 3\xi^1_{\ ,u}u_{,1}\right) = 0. \qquad (99)
\end{aligned}$$

Note that $H_A = 0$ eliminates some terms in the last two parts of the above equation, giving that $2\xi^1_{\ ,1} = \xi^2_{\ ,2}$. We can easily read off more conditions from above, giving the system of partial differential equations that we need to solve to be:

$$\begin{aligned}
\xi^2_{\ ,1} = 0, &\qquad \xi^2_{\ ,u} = 0, \\
\xi^1_{\ ,u} = 0, &\qquad \eta_{,uu} = 0, \\
\eta_{,11} = \eta_{,2}, &\qquad 2\eta_{,1u} - \xi^1_{,11} + \xi^1_{\ ,2} = 0, \\
\xi^2_{\ ,2} = 2\xi^1_{\ ,1}. & \qquad (100)
\end{aligned}$$

By the first and last conditions above, $\xi^1_{\ ,11} = 0$, so

$$\xi^1 = xe(t) + f(t), \qquad (101)$$

$$2\eta_{,1u} + xe'(t) + f'(t) = 0. \qquad (102)$$

Since $\eta_{,uu} = 0$ this implies that $\eta_{,1u} = z_{,1}(x,t)$, and so by (102),

$$\eta_{,u} = z(x,t) = -\frac{1}{4}x^2e'(t) - \frac{1}{2}xf'(t) - g(t), \qquad (103)$$

$$\eta = -\frac{1}{4}ux^2e'(t) - \frac{1}{2}uxf'(t) - ug(t) + h(x,t). \qquad (104)$$

By $\eta_{,11} = \eta_{,2}$ we have

$$\frac{1}{2}ue'(t) + h_{,xx} = \frac{1}{4}ux^2e''(t) + \frac{1}{2}uxf''(t) + ug'(t) + h_{,t}, \tag{105}$$

so that $e''(t) = f''(t) = 0$, $g'(t) = \frac{1}{2}e'(t)$, and thus

$$\begin{aligned} \xi^1 &= a_1xt + a_2x + a_3t + a_4, \\ \xi^2 &= a_1t^2 + 2a_2t + a_5, \\ \eta &= -\frac{1}{4}a_1ux^2 - \frac{1}{2}a_3ux - \frac{1}{2}a_1ut + a_6u + h(x,t), \end{aligned} \tag{106}$$

where the a_i are arbitrary constants and $h(x,t)$ is any solution to the heat equation. The point transformation generated will be $\tilde{u} = u + \varepsilon h(x,t) + ...$; this simply tells us that a linear combination of any solutions to the heat equation are themselves solutions (which is an obvious feature of theirs). We do not learn anything more from this; since $h(x,t)$ is arbitrary, we will set it equal to 0 and proceed.

And so our generator is

$$\begin{aligned} \mathbf{X} &= (a_1xt + a_2x + a_3t + a_4)\frac{\partial}{\partial x} + \left(a_1t^2 + 2a_2t + a_5\right)\frac{\partial}{\partial t} \\ &+\left(-\frac{1}{2}a_3ux - a_1(\frac{1}{2}ut + \frac{1}{4}ux^2) + a_6u + h(x,t)\right)\frac{\partial}{\partial u} \end{aligned} \tag{107}$$

Now we consider what sort of transformations these are by writing X as a six-parameter family (one for each constant a_i):

$$\begin{aligned} \mathbf{X_1} &= xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{1}{2}ut + \frac{1}{4}ux^2\right)\frac{\partial}{\partial u}, \\ \mathbf{X_2} &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\ \mathbf{X_3} &= t\frac{\partial}{\partial x} - \frac{1}{2}ux\frac{\partial}{\partial u}, \\ \mathbf{X_4} &= \frac{\partial}{\partial x}, \quad \mathbf{X_5} = \frac{\partial}{\partial t}, \quad \mathbf{X_6} = u\frac{\partial}{\partial u}. \end{aligned} \tag{108}$$

What sort of transformations do these operators correspond to? From the previous subsection, we can calculate the finite symmetry transformations, from which we quickly see that $\mathbf{X_4}$ and $\mathbf{X_5}$ are translations, and $\mathbf{X_6}$ and $\mathbf{X_2}$

are similarity (scaling) transformations. For $\mathbf{X_3}$ we have

$$\frac{d\tilde{x}}{d\varepsilon} = \tilde{t}, \qquad \frac{d\tilde{t}}{d\varepsilon} = 0, \qquad \frac{d\tilde{u}}{d\varepsilon} = -\frac{1}{2}\tilde{u}\tilde{x}, \tag{109}$$

so that

$$\tilde{t} = t, \qquad \tilde{x} = \varepsilon t + x, \qquad \tilde{u} = ue^{-\frac{1}{2}\varepsilon(x+\frac{1}{2}\varepsilon t)}, \tag{110}$$

which is a Galilean transformation for x ($x = \tilde{x} + vt$). Inserting a known solution $\tilde{u} = A =$ constant gives us a new solution to the heat equation

$$u = Ae^{\frac{1}{2}\varepsilon(x+\frac{1}{2}t\varepsilon)} \tag{111}$$

Using the same method on $\mathbf{X_1}$:

$$\mathbf{X}_1 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{1}{2}ut + \frac{1}{4}ux^2\right)\frac{\partial}{\partial u}, \tag{112}$$

giving

$$\frac{d\tilde{x}}{d\varepsilon} = \tilde{x}\tilde{t}, \qquad \frac{d\tilde{t}}{d\varepsilon} = \tilde{t}^2, \qquad \frac{d\tilde{u}}{d\varepsilon} = -\tilde{u}\left(\frac{\tilde{x}^2}{4} + \frac{\tilde{t}}{2}\right). \tag{113}$$

Stephani calculates $\tilde{x}, \tilde{t}$ and $\tilde{u}$ in his book, and it is not difficult to do so using the initial conditions $\tilde{x}|_{\varepsilon=0} = x$ etc.:

$$\tilde{x} = \frac{x}{(1-\varepsilon t)}, \qquad \tilde{t} = \frac{t}{(1-\varepsilon t)},$$

$$\tilde{u} = u\sqrt{1-\varepsilon t}e^{-\varepsilon x^2/4(1-\varepsilon t)}. \tag{114}$$

If we insert our known solution $\tilde{u} = A =$ constant, we have a new solution

$$u = \frac{A}{\sqrt{1-\varepsilon t}}\, e^{\varepsilon x^2/4(1-\varepsilon t)}, \tag{115}$$

which looks similar to one we'd get by considering similarity solutions of the original equation – in fact, we'd know to try solutions of the form $f(\frac{-x^2}{t})$ because it is a symmetry given by $\mathbf{X}_2$. Since ε is supposed to be small, this new solution is very close to $u = A$ (as you'd expect from a finite symmetry transform) and looks like an attempt at a boundary layer solution at $\hat{t} = 1$. And of course, the previous solution (111) would also work if we substituted it in here!

Recall that we can use the same method on linear combinations of the symmetry generators. For example,

$$\mathbf{X_2} + \mathbf{X_3} = (x+t)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \frac{1}{2}ux\frac{\partial}{\partial u} \tag{116}$$

has

$$\frac{d\tilde{x}}{d\varepsilon} = \tilde{x} + \tilde{t}, \qquad \frac{d\tilde{t}}{d\varepsilon} = 2\tilde{t}, \qquad \frac{d\tilde{u}}{d\varepsilon} = -\frac{1}{2}\tilde{u}\tilde{x}, \tag{117}$$

with solution

$$\begin{gathered} \tilde{t} = te^{2\varepsilon}, \quad \tilde{x} = te^{2\varepsilon} + (x-t)e^{\varepsilon}, \\ \tilde{u} = u \exp\left[\tfrac{1}{2}\, x(1-e^{\varepsilon}) - \tfrac{1}{4}\, t\left(1 + e^{2\varepsilon} - 2e^{\varepsilon}\right)\right]. \end{gathered} \tag{118}$$

Finally, the method described at the end of the last subsection should also let us reduce this into a first order equation, but this essay shall not bother to do so.

3.4 Symmetries of the time-dependent SN equations

Before we start, we note that the SN equations contain terms involving $|\psi|^2 = \psi\overline{\psi}$; although at first it seems like we would increase our workload by considering $u^2 = \overline{\psi}$ to be another dependent variable, doing such a thing will probably be neater and safer than playing around with the moduli of functions, as long as we remember that (and restrict) this new variable to be the complex conjugate of $u^1 = \psi$. If this is the case, then note that the following holds for η^u and η^v (with $\varepsilon \in \mathbb{R}$):

$$\eta^v = \eta^{\overline{u}} = \left.\frac{\partial \overline{\tilde{u}}}{\partial \varepsilon}\right|_{\varepsilon=0} = \overline{\eta^u}. \tag{119}$$

Since our co-ordinates x^i must remain real (it would not make any physical sense otherwise) the functions ξ^i should also be real, unlike η^u, η^v. We should also expect η^w to be real as well, given its form (19).

If we start by writing our variables as

$$\begin{gathered} x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \\ u^1 = u = \psi, \quad u^2 = v = \overline{\psi}, \quad u^3 = w = \phi, \end{gathered} \tag{120}$$

then the non-dimensionalised, free, time-dependent Schrödinger-Newton equations are given by

$$H_1 = iu_{,0} + u_{,11} + u_{,22} + u_{,33} - uw = 0 \tag{121}$$

$$H_2 = iv_{,0} - v_{,11} - v_{,22} - v_{,33} + vw = 0 \tag{122}$$

$$H_3 = w_{,11} + w_{,22} + w_{,33} - uv = 0. \tag{123}$$

It is obvious that this equation is invariant under co-ordinate translations, since $x^\alpha \mapsto x^\alpha + \varepsilon$ doesn't change their form. We should also expect to see spatial rotations (as no 'preferred' axis is given) and rescalings.

Considering a generator $\mathbf{X}$ such that each $\mathbf{X}H_j = 0$, equation (72) gives us our symmetry conditions:

$$\mathbf{X}H_1 = i\eta^u_0 + \eta^u_{11} + \eta^u_{22} + \eta^u_{33} - \eta^u w - \eta^w u = 0, \tag{124}$$

$$\mathbf{X}H_2 = i\eta^v_0 - \eta^v_{11} - \eta^v_{22} - \eta^v_{33} + \eta^v w + \eta^w v = 0, \tag{125}$$

$$\mathbf{X}H_3 = \eta^w_{11} + \eta^w_{22} + \eta^w_{33} - \eta^u v - \eta^v u = 0. \tag{126}$$

We now compute for each of the η^α_{nm} in the above system the terms that contain second derivatives $u^\alpha_{,nm}$ that don't occur in each $\mathbf{X}H_\alpha = 0$ equation using relation (77):

$$\begin{aligned}
\eta^\alpha_{nn} &= -2\xi^i_{,n}u^\alpha_{,ni} - 2\xi^k_{,\beta}u^\beta_{,n}u^\alpha_{,nk}, \\
\eta^\alpha_{11} &= -2(\xi^0_{,1}u^\alpha_{,10} + \xi^2_{,1}u^\alpha_{,12} + \xi^3_{,1}u^\alpha_{,13} \\
&\quad + \xi^0_{,u}u_{,1}u^\alpha_{,10} + \xi^2_{,u}u_{,1}u^\alpha_{,12} + \xi^3_{,u}u_{,1}u^\alpha_{,13} \\
&\quad + \xi^0_{,v}v_{,1}u^\alpha_{,10} + \xi^2_{,v}v_{,1}u^\alpha_{,12} + \xi^3_{,v}v_{,1}u^\alpha_{,13} \\
&\quad + \xi^0_{,w}w_{,1}u^\alpha_{,10} + \xi^2_{,w}w_{,1}u^\alpha_{,12} + \xi^3_{,w}w_{,1}u^\alpha_{,13}), \qquad (127) \\
\eta^\alpha_{22} &= -2(\xi^0_{,2}u_{,20} + \xi^1_{,2}u_{,21} + \xi^3_{,2}u_{,23} \\
&\quad + \xi^0_{,u}u_{,2}u_{,20} + \xi^1_{,u}u_{,2}u_{,21} + \xi^3_{,u}u_{,2}u_{,23} \\
&\quad + \xi^0_{,v}v_{,2}u_{,20} + \xi^1_{,v}v_{,2}u_{,21} + \xi^3_{,v}v_{,2}u_{,23} \\
&\quad + \xi^0_{,w}w_{,2}u_{,20} + \xi^1_{,w}w_{,2}u_{,21} + \xi^3_{,w}w_{,2}u_{,23}), \qquad (128) \\
\eta^\alpha_{33} &= -2(\xi^0_{,3}u_{,30} + \xi^1_{,3}u_{,31} + \xi^2_{,3}u_{,32} \\
&\quad + \xi^0_{,u}u_{,3}u_{,30} + \xi^1_{,u}u_{,3}u_{,31} + \xi^2_{,u}u_{,3}u_{,32} \\
&\quad + \xi^0_{,v}v_{,3}u_{,30} + \xi^1_{,v}v_{,3}u_{,31} + \xi^2_{,v}v_{,3}u_{,32} \\
&\quad + \xi^0_{,w}w_{,3}u_{,30} + \xi^1_{,w}w_{,3}u_{,31} + \xi^2_{,w}w_{,3}u_{,32}). \qquad (129)
\end{aligned}$$

Our symmetry conditions are, to second order, $\eta^\alpha_{11} + \eta^\alpha_{22} + \eta^\alpha_{33} + ... = 0$, which gives the following conditions on ξ:

$$\xi^i_{,\beta} = 0, \qquad (130)$$

$$\xi^0_{,j} = 0, \qquad (131)$$

$$\xi^1_{,2} = -\xi^2_{,1},\ \xi^1_{,3} = -\xi^3_{,1},\ \xi^2_{,3} = -\xi^3_{,2}, \qquad (132)$$

where $i = 0, 1, 2, 3$ and $j = 1, 2, 3$ (note that where I use i and j as subscripts in future, they are summed over these same ranges).

Now we shall calculate all second order derivatives for η^α_{nn}, and apply the above conditions:

$$\begin{aligned}
\eta^u_{11} &= \eta^u_{,u}u_{,11} + \eta^u_{,v}v_{,11} + \eta^u_{,w}w_{,11} - 2\xi^1_{,1}u_{,11}, \qquad (133) \\
\eta^u_{22} &= \eta^u_{,u}u_{,22} + \eta^u_{,v}v_{,22} + \eta^u_{,w}w_{,22} - 2\xi^2_{,2}u_{,22}, \qquad (134) \\
\eta^u_{33} &= \eta^u_{,u}u_{,33} + \eta^u_{,v}v_{,33} + \eta^u_{,w}w_{,33} - 2\xi^3_{,3}u_{,33}. \qquad (135)
\end{aligned}$$

To get more conditions, insert these into $\mathbf{X}H_1 = 0$, again considered to only second order, and using $H_1 = 0$ also to second order, i.e. $H_1 = u_{,11} + u_{,22} + u_{,33} + ... = 0$:

$$\begin{aligned}
\mathbf{X}H_1 &= \eta^{\alpha}_{11} + \eta^{\alpha}_{22} + \eta^{\alpha}_{33} + ... \\
&= \left(\eta^u_u + \eta^u_v + \eta^u_w\right)\left(u_{,11} + u_{,22} + u_{,33}\right) \\
&\quad -2\left(\xi^1_1 u_{,11} + \xi^2_{,2} u_{,22} + \xi^3_{,3} u_{,33}\right) + ... \\
&= 0
\end{aligned} \tag{136}$$

which gives us the condition

$$\xi^1_{,1} = \xi^2_{,2} = \xi^3_{,3}. \tag{137}$$

Using this and (132):

$$\begin{aligned}
\xi^1_{,11} = -\xi^1_{,22} = -\xi^1_{,33}, \\
-\xi^2_{,11} = \xi^2_{,22} = -\xi^2_{,33}, \\
-\xi^3_{,11} = -\xi^3_{,22} = \xi^3_{,33}.
\end{aligned} \tag{138}$$

It should be noted that up to this point we'd get the exact same conditions by considering $\mathbf{X}H_2$ or $\mathbf{X}H_3$ as they are similar at second order.

Now we write out symmetry conditions out fully and apply conditions:

$$\begin{aligned}
\mathbf{X}H_1 &= i\eta^u_0 + \eta^u_{11} + \eta^u_{22} + \eta^u_{33} - \eta^u w - \eta^w u \\
&= i(\eta^u_{,0} + \eta^u_{,u} u_{,0} + \eta^u_{,v} v_{,0} + \eta^u_{,w} w_{,0} \\
&\quad -\xi^0_{,0} u_{,0} - \xi^1_{,0} u_{,1} - \xi^2_{,0} u_{,2} - \xi^3_{,0} u_{,3}) \\
&\quad + \eta^u_{,11} + \eta^u_{,22} + \eta^u_{,33} - \eta^u w - \eta^w u \\
&\quad + 2(\eta^u_{,1u} u_{,1} + \eta^u_{,1v} v_{,1} + \eta^u_{,1w} w_{,1} + \eta^u_{,2u} u_{,2} \\
&\quad + \eta^u_{,2v} v_{,2} + \eta^u_{,2w} w_{,2} + \eta^u_{,3u} u_{,3} + \eta^u_{,3v} v_{,3} \\
&\quad + \eta^u_{,3w} w_{,3}) - \xi^1_{,11} u_{,1} - \xi^2_{,11} u_{,2} - \xi^3_{,11} u_{,3} \\
&\quad - \xi^1_{,22} u_{,1} - \xi^2_{,22} u_{,2} - \xi^3_{,22} u_{,3} - \xi^1_{,33} u_{,1} \\
&\quad - \xi^2_{,33} u_{,2} - \xi^3_{,33} u_{,3} + \eta^u_{,uu}\left(u_{,1}^2 + u_{,2}^2 + u_{,3}^2\right) \\
&\quad + \eta^u_{,vv}\left(v_{,1}^2 + v_{,2}^2 + v_{,3}^2\right) + \eta^u_{,ww}\left(w_{,1}^2 + w_{,2}^2 + w_{,3}^2\right) \\
&\quad + \eta^u_{,uv}\left(u_{,1} v_{,1} + u_{,2} v_{,2} + u_{,3} v_{,3}\right) \\
&\quad + \eta^u_{,uw}\left(u_{,1} w_{,1} + u_{,2} w_{,2} + u_{,3} w_{,3}\right)
\end{aligned}$$

$$
\begin{aligned}
& + \eta^u_{,vw}\left(v_{,1}w_{,1} + v_{,2}w_{,2} + v_{,3}w_{,3}\right) \\
& + \eta^u_{,u}u_{,11} + \eta^u_{,v}v_{,11} + \eta^u_{,w}w_{,11} \\
& + \eta^u_{,u}u_{,22} + \eta^u_{,v}v_{,22} + \eta^u_{,w}w_{,22} \\
& + \eta^u_{,u}u_{,33} + \eta^u_{,v}v_{,33} + \eta^u_{,w}w_{,33} \\
& - 2\xi^1_{,1}u_{,11} - 2\xi^2_{,2}u_{,22} - 2\xi^3_{,3}u_{,33} \\
= \; & 0. \qquad (139)
\end{aligned}
$$

Equation (139) must be satisfied identically in the derivatives of u^α, so some conditions can be quickly read off:

$$\eta^u_{,v} = \eta^u_{,w} = 0, \qquad (140)$$

$$\eta^u_{,\alpha\beta} = 0, \qquad (141)$$

$$-i\xi^j_{,0} + 2\eta^u_{,ju} - \xi^j_{,11} - \xi^j_{,22} - \xi^j_{,33} = 0, \qquad (142)$$

for $j = 1, 2, 3$. If we focus on the remaining terms in this equation,

$$
\begin{aligned}
iu_{,0}\left(\eta^u_{,u} - \xi^0_{,0}\right) + u_{,11}\left(\eta^u_{,u} - 2\xi^1_{,1}\right) & \\
+u_{,22}\left(\eta^u_{,u} - 2\xi^2_{,2}\right) + u_{,33}\left(\eta^u_{,u} - 2\xi^3_{,3}\right) & \\
+\eta^u_{,11} + \eta^u_{,22} + \eta^u_{,33} - \eta^u w - \eta^w u + i\eta^u_{,0} & = 0. \qquad (143)
\end{aligned}
$$

Using $H_1 = 0$, the terms with $\eta^u_{,u}$ can be replaced and this equation simplifies to

$$
\begin{aligned}
i\xi^0_{,0}u_{,0} + 2\xi^1_{,1}u_{,11} + 2\xi^2_{,2}u_{,22} + 2\xi^3_{,3}u_{,33} - \eta^u_{,u}uw & \\
+\eta^u w + \eta^w u - \eta^u_{,11} - \eta^u_{,22} - \eta^u_{,33} - i\eta^u_{,0} & = 0, \qquad (144)
\end{aligned}
$$

and hence

$$\xi^0_{,0} = 2\xi^1_{,1} = 2\xi^2_{,2} = 2\xi^3_{,3}, \qquad (145)$$

leaving us with

$$(\xi^0_{,0} - \eta^u_{,u})uw + \eta^u w + \eta^w u - \nabla^2\eta^u - i\eta^u_{,0} = 0. \qquad (146)$$

We will come back to sort this equation out after considering the other two symmetries. However, conditions (140) and (141) give

$$\eta^u = u\,U(t,x,y,z) + \tilde{U}(t,x,y,z). \tag{147}$$

Now, since ξ^0 is a function of t only, by (145) the ξ^j are only to first order in x^j:

$$\begin{aligned} \xi^0 &= e(t), \\ 2\xi^1 &= xe'(t) + \tilde{X}(t,y,z), \\ 2\xi^2 &= ye'(t) + \tilde{Y}(t,x,z), \\ 2\xi^3 &= ze'(t) + \tilde{Z}(t,x,y), \end{aligned} \tag{148}$$

so (142) is reduced to

$$i\xi^j_{,0} = 2\eta^u_{,ju}. \tag{149}$$

Note that ξ^i satisfies the conformal Killing vector equation:

$$\xi^i_{,j} + \xi^j_{,i} = e'(t)\,\delta_{ij} \;\; : i,j \neq 0. \tag{150}$$

To solve, use conditions (132) on (148) and integrate to get:

$$\begin{aligned} 2\xi^1 &= e'(t)x + a_2(t)y + a_3(t)z + a_6 + 2X(t), \\ 2\xi^2 &= -a_2(t)x + e'(t)y + a_4(t)z + a_7 + 2Y(t), \\ 2\xi^3 &= -a_3(t)x - a_4(t)y + e'(t)z + a_8 + 2Z(t), \end{aligned} \tag{151}$$

where the functions of time and the constants a_k are as yet unknown, but real valued (by the argument given at the beginning of this chapter).

Now we consider the other two symmetry conditions, taking into account conditions on ξ we have found above. The same process with $\mathbf{X}H_2$ will give

$$i\xi^j_{,0} = -2\eta^v_{,jv}, \tag{152}$$

$$(\eta^v_{,v} - \xi^0_{,0})vw + \nabla^2\eta^v - i\eta^v_{,0} - \eta^v w - \eta^w v = 0, \tag{153}$$

$$\eta^v = v\,V(t,x,y,z) + \tilde{V}(t,x,y,z). \tag{154}$$

And for $\mathbf{X}H_3$, in which time doesn't appear,

$$\begin{aligned}
\mathbf{X}H_3 &= \eta^w_{11} + \eta^w_{22} + \eta^w_{33} - \eta^u v - \eta^v u \\
&= \eta^w_{,11} + \eta^w_{,22} + \eta^w_{,33} - \eta^u v - \eta^v u \\
&\quad +2(\eta^w_{,1u} u_{,1} + \eta^w_{,1v} v_{,1} + \eta^w_{,1w} w_{,1} + \eta^w_{,2u} u_{,2} \\
&\quad + \eta^w_{,2v} v_{,2} + \eta^w_{,2w} w_{,2} + \eta^w_{,3u} u_{,3} + \eta^w_{,3v} v_{,3} \\
&\quad + \eta^w_{,3w} w_{,3}) + \eta^w_{,u} u_{,11} + \eta^w_{,v} v_{,11} + \eta^w_{,w} w_{,11} \\
&\quad + \eta^w_{,u} u_{,22} + \eta^w_{,v} v_{,22} + \eta^w_{,w} w_{,22} + \eta^w_{,u} u_{,33} \\
&\quad + \eta^w_{,v} v_{,33} + \eta^w_{,w} w_{,33} - 2\xi^1_{,1} w_{,11} - 2\xi^2_{,2} w_{,22} \\
&\quad - 2\xi^3_{,3} w_{,33} \\
&= 0.
\end{aligned} \tag{155}$$

This gives

$$\eta^w_{,ww} = \eta^w_{,u} = \eta^w_{,v} = 0, \tag{156}$$

$$\eta^w_{,jw} = 0, \tag{157}$$

$$\left(\eta^w_{,w} - \xi^0_{,0}\right) uv + \nabla^2 \eta^w - \eta^u v - \eta^v u = 0, \tag{158}$$

with

$$\eta^w = w\, W(t) + \tilde{W}(t, x, y, z). \tag{159}$$

We will now insert this, (147) and (154) into equations (146), (153) and (158) to calculate the η^α and ξ^i. Starting with (146):

$$uw(e'(t) + W) + u(\tilde{W} - \nabla^2 U - iU_{,0}) + w\tilde{U} - \nabla^2 \tilde{U} - i\tilde{U}_{,0} = 0, \tag{160}$$

giving

$$\tilde{U} = 0, \quad W = -e'(t), \quad \tilde{W} = \nabla^2 U + iU_{,0}. \tag{161}$$

Next,

$$vw(V - e'(t)) + v(\nabla^2 V - iV_{,0} - \tilde{W}) - w\tilde{V} + \nabla^2 \tilde{V} + i\tilde{V}_{,0} = 0, \tag{162}$$

giving

$$\tilde{V} = 0, \quad \tilde{W} = \nabla^2 V - iV_{,0}. \tag{163}$$

And finally

$$uv(W - e'(t)) + \nabla^2 \tilde{W} - uvU - uvV = 0, \tag{164}$$

giving

$$U + V = -2e'(t), \quad \nabla^2 \tilde{W} = 0. \tag{165}$$

If we add together (161) and (163), and use (165),

$$\begin{aligned} 2\tilde{W} &= i(U_{,0} - V_{,0}) \\ &= 2i[U_{,0} + e''(t)] \\ \text{so } \nabla^2 \tilde{W} &= i\nabla^2[U_{,0} + e''(t)] = 0 \\ \text{and } \tilde{W}_{,0} &= i[U_{,00} + e'''(t)]. \end{aligned} \tag{166}$$

But differentiating (161) and using $\nabla^2 U_{,0} = 0$ from the above has that

$$\tilde{W}_{,0} = iU_{,00}, \tag{167}$$

and so by comparison $e'''(t) = 0$, and we shall write

$$\xi^0 = e(t) = a_0 t^2 + 2a_1 t + a_5. \tag{168}$$

By (145) and (149),

$$i\xi^0_{\ ,00} = 2ia_0 = 2i\xi^j_{\ ,0j} = 4\eta^u_{\ ,jju} = 4U_{,jj}, \tag{169}$$

and similarly for V; with $4U + 4V = -8e'(t)$ this gives

$$\begin{aligned} 4U &= \frac{1}{2} ia_0(x^2 + y^2 + z^2) + x\hat{U}_1(t) + y\hat{U}_2(t) + z\hat{U}_3(t) \\ &\quad + \hat{U}_4(t) + xy\hat{U}_6(t) + xz\hat{U}_7(t) + yz\hat{U}_8 + xyz\hat{U}_9(t), \\ 4V &= -\frac{1}{2} ia_0(x^2 + y^2 + z^2) - x\hat{U}_1(t) - y\hat{U}_2(t) - z\hat{U}_3(t) \\ &\quad + \hat{U}_5(t) - xy\hat{U}_6(t) - xz\hat{U}_7(t) - yz\hat{U}_8 - xyz\hat{U}_9(t), \end{aligned} \tag{170}$$

with

$$\hat{U}_4(t) + \hat{U}_5(t) = -8e'(t) = -16a_0 t - 16a_1. \tag{171}$$

Note that these functions $\hat{U}_k(t)$ are possibly complex. If we insert these into

(161) and (163) to find $\tilde{W}$,

$$\begin{aligned} 4\tilde{W} &= 3ia_0 + i\left[x\hat{U}_1' + y\hat{U}_2' + z\hat{U}_3' + \hat{U}_4' + ...\right] \\ &= -3ia_0 + i\left[x\hat{U}_1' + y\hat{U}_2' + z\hat{U}_3' - \hat{U}_5' + ...\right], \end{aligned} \tag{172}$$

and thus through equality,

$$6ia_0 + i\left[\hat{U}_4' + \hat{U}_5'\right] = 6ia_0 - 16ia_0 = 0, \tag{173}$$

giving $a_0 = 0$. This helps us find the form of ξ^i, so insert the above into condition (149):

$$\begin{aligned} 2i\xi^j_{,0} \;=\; 4\eta^u_{,ju} &= 4U_{,j}(x,y,z); \\ i\left[a_2'y + a_3'z + 2X'\right] &= \hat{U}_1 + y\hat{U}_6 + z\hat{U}_7 + yz\hat{U}_9, \\ i\left[-a_2'x + a_4'z + 2Y'\right] &= \hat{U}_2 + x\hat{U}_6 + z\hat{U}_8 + xz\hat{U}_9, \\ i\left[-a_3'x - a_4'y + 2Z'\right] &= \hat{U}_3 + x\hat{U}_7 + y\hat{U}_8 + xy\hat{U}_9. \end{aligned} \tag{174}$$

From this we can calculate the functions $\hat{U}_k(t)$:

$$\begin{gathered} \hat{U}_1(t) = 2iX'(t), \quad \hat{U}_2(t) = 2iY'(t), \quad \hat{U}_3(t) = 2iZ'(t), \\ \hat{U}_6(t) = 0 = a_2'(t), \quad \hat{U}_7(t) = 0 = a_3'(t), \quad \hat{U}_8(t) = 0 = a_4'(t), \\ \hat{U}_9(t) = 0. \end{gathered} \tag{175}$$

Finally, recall that v is the complex conjugate of u and must remain so under a transformation, with $\eta^v \equiv \overline{\eta^u}$. Therefore, $\hat{U}_4(t) = -8a_1 + 4i\Omega(t)$ and $\hat{U}_5(t) = -8a_1 - 4i\Omega(t)$ (where $\Omega(t)$ is arbitrary). If we also redefine $a_2(t) = 2a_2 =$ constant etc., and set $\mathbf{T} = (X(t), Y(t), Z(t))$, then

$$\begin{aligned} \xi^0 &= 2a_1t + a_5, \\ \xi^1 &= a_1x + a_2y + a_3z + a_6 + X(t), \\ \xi^2 &= -a_2x + a_1y + a_4z + a_7 + Y(t), \\ \xi^3 &= -a_3x - a_4y + a_1z + a_8 + Z(t), \end{aligned} \tag{176}$$

$$\eta^u = u\left[\frac{1}{2}ix^j\mathbf{T}_j' - 2a_1 + i\Omega(t)\right],$$

$$\begin{aligned}
\eta^v &= -v\left[\frac{1}{2}ix^j\mathbf{T}'_j + 2a_1 + i\Omega(t)\right], \\
\eta^w &= -2a_1 w - \frac{1}{2}\,x^j\mathbf{T}''_j - \Omega'(t).
\end{aligned} \tag{177}$$

And at last we've calculated our symmetry generator!

$$\begin{aligned}
\mathbf{X} = \; & [2a_1 t + a_5]\frac{\partial}{\partial t} + [a_1 x + a_2 y + a_3 z + a_6 + X(t)]\frac{\partial}{\partial x} \\
& + [-a_2 x + a_1 y + a_4 z + a_7 + Y(t)]\frac{\partial}{\partial y} \\
& + [-a_3 x - a_4 y + a_1 z + a_8 + Z(t)]\frac{\partial}{\partial z} \\
& + u\left[\frac{1}{2}ix^j\mathbf{T}'_j - 2a_1 + i\Omega(t)\right]\frac{\partial}{\partial u} \\
& - v\left[\frac{1}{2}ix^j\mathbf{T}'_j + 2a_1 + i\Omega(t)\right]\frac{\partial}{\partial v} \\
& + \left[-2a_1 w - \frac{1}{2}\,x^j\mathbf{T}''_j - \Omega'(t)\right]\frac{\partial}{\partial w}
\end{aligned} \tag{178}$$

This gives the following Lie algebra:

$$\begin{aligned}
\mathbf{X_1} &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} - 2w\frac{\partial}{\partial w}, \\
\mathbf{X_2} &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\
\mathbf{X_3} &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \\
\mathbf{X_4} &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\
\mathbf{X_5} &= \frac{\partial}{\partial t}, \qquad \mathbf{X_6} = \frac{\partial}{\partial x}, \qquad \mathbf{X_7} = \frac{\partial}{\partial y}, \qquad \mathbf{X_8} = \frac{\partial}{\partial z}, \\
\mathbf{X_9} &= \mathbf{T}_j\frac{\partial}{\partial x^j} + \frac{i}{2}x^j\left[\mathbf{T}'_j\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right) + i\,\mathbf{T}''_j\frac{\partial}{\partial w}\right], \\
\mathbf{X_{10}} &= i\Omega(t)\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right) - \Omega'(t)\frac{\partial}{\partial w}.
\end{aligned} \tag{179}$$

Note that we could instead write $ia_{10}\Omega(t)$ and so on for these arbitrary functions of time, hence why we have included them as forming separate generators.

3.4.1 Generators: $\mathbf{X_1}$ to $\mathbf{X_8}$

As to the type of transformations these correspond to, $\mathbf{X_5}$, $\mathbf{X_6}$, $\mathbf{X_7}$ and $\mathbf{X_8}$ are obviously translations, and we note that $\mathbf{X_2}, \mathbf{X_3}$ and $\mathbf{X_4}$ commute, with their Lie brackets given by

$$[\mathbf{X_2}, \mathbf{X_3}] = \mathbf{X_4}, \quad [\mathbf{X_4}, \mathbf{X_2}] = \mathbf{X_3}, \quad [\mathbf{X_3}, \mathbf{X_4}] = \mathbf{X_2}, \tag{180}$$

and they are obviously rotations. And by comparison with the heat equation example, $\mathbf{X_1}$ is a scaling given by

$$\tilde{t} = te^{2\varepsilon}, \qquad \tilde{x}^j = x^j e^{\varepsilon}, \qquad \tilde{u}^{\alpha} = u^{\alpha} e^{-2\varepsilon}. \tag{181}$$

3.4.2 Generators: $\mathbf{X_9}$

At this point, the reader may be wondering how we deal with these remaining arbitrary time functions. Remember that a linear combination of any of these infinitesimal generators is also a symmetry generator, so we can proceed as normal, noting the lack of a $\frac{\partial}{\partial t}$ term in $\mathbf{X_9}$ makes things slightly easier:

$$\begin{aligned}
\tilde{x}^j &= x^j + \varepsilon \mathbf{T}^j, \quad \tilde{t} = t, \\
\tilde{u} &= u \exp\left[\frac{1}{4} i \varepsilon \mathbf{T}'_j \left(2x^j + \varepsilon \mathbf{T}^j\right)\right], \quad \tilde{v} = \overline{\tilde{u}} = ... \\
\tilde{w} &= w - \frac{1}{4} \varepsilon \mathbf{T}''_j \left(2x^j + \varepsilon \mathbf{T}^j\right).
\end{aligned} \tag{182}$$

This is another Galilean transformation, of sorts, and describes what will happen to ψ, ϕ in a moving frame. Under acceleration, ψ will undergo a phase shift, and the self-potential ϕ will also change. If we redefine $\mathbf{S}(t) = \varepsilon \mathbf{T}$ and ignore $O(\varepsilon^2)$, then we can rewrite this as

$$\begin{aligned}
\tilde{x}^j &= x^j + \mathbf{S}^j, \quad \tilde{t} = t, \\
\tilde{u} &= u \exp\left[\frac{1}{2} i x^j \mathbf{S}'_j\right], \quad \tilde{v} = \overline{\tilde{u}} = ... \\
\tilde{w} &= w - \frac{1}{2} x^j \mathbf{S}''_j.
\end{aligned} \tag{183}$$

Motion, for example, that is confined to accelerating in the x direction at a constant rate, where $\mathbf{T} = (t^2, 0, 0)$, has

$$\begin{gathered}
\tilde{x} = x + \varepsilon t^2, \quad \tilde{y} = y, \quad \tilde{z} = z, \quad \tilde{t} = t, \\
\tilde{u} = u \exp\left[\tfrac{1}{2} i \varepsilon t \left(2x + \varepsilon t^2\right)\right], \\
\tilde{w} = w - \tfrac{1}{2}\varepsilon(2x + \varepsilon t^2) = w - \varepsilon \tilde{x} + O(\varepsilon^2).
\end{gathered} \tag{184}$$

However, we do not have any change in the potential ϕ in a non-accelerating frame. But note the connection with the equivalence principle, since $\tilde{w}$ represents a gravitational potential!

Keeping the same form for $\mathbf{T}$, we will use this symmetry to find a similarity variable and reduce the system by the scheme described in the last subsection. We start by taking the normal variable to be

$$\chi = \varepsilon = \frac{\tilde{x} - x}{t^2}, \tag{185}$$

which we insert into (184) to get

$$\begin{aligned}
\tilde{y} &= y, \text{ etc.}, \\
\tilde{u} &= u \exp\left(i\, \frac{\tilde{x}^2 - x^2}{2t}\right), \quad \tilde{v} = \overline{\tilde{u}} = ..., \\
\tilde{w} &= w - \frac{\tilde{x}^2 - x^2}{2t^2}.
\end{aligned} \tag{186}$$

From this, we can read off the similarity variables:

$$\begin{aligned}
\varphi^1 &= y, \quad \varphi^2 = z, \quad \varphi^3 = t, \\
\varphi^4 &= u \exp\left(\frac{-ix^2}{2t}\right), \quad \varphi^5 = v \exp\left(\frac{ix^2}{2t}\right), \\
\varphi^6 &= w + \frac{x^2}{2t^2}.
\end{aligned} \tag{187}$$

Take $\varphi^4, \varphi^5, \varphi^6$ as new dependent variables w^α and the others as independent variables y^k. Rearrange so that $u^\alpha = w^\alpha f^\alpha(x,t)$ first:

$$u = w^u \exp\left(\frac{ix^2}{2t}\right), \quad v = w^v \exp\left(\frac{-ix^2}{2t}\right), \quad w = w^w - \frac{x^2}{2t^2}. \tag{188}$$

Now use the chain rule to get

$$\begin{aligned}
u_{,t} &= \frac{\partial u}{\partial w^\beta}\frac{\partial w^\beta}{\partial y^k}\frac{\partial y^k}{\partial t} + \frac{\partial u}{\partial y^k}\frac{\partial y^k}{\partial t} + \frac{\partial u}{\partial \chi}\frac{\partial \chi}{\partial t} \\
&= \left(\frac{\partial w^u}{\partial y^3} - i\frac{w^u x^2}{2t^2}\right)\exp\left(\frac{ix^2}{2t}\right), \\
u_{,xx} &= \left(\frac{i}{t} - \frac{x^2}{t^2}\right) w^u \exp\left(\frac{ix^2}{2t}\right), \\
u_{,yy} &= \frac{\partial}{\partial y^1}\left(\frac{\partial w^u}{\partial y^1}\right)\exp\left(\frac{ix^2}{2t}\right), \\
u_{,zz} &= \frac{\partial}{\partial y^2}\left(\frac{\partial w^u}{\partial y^2}\right)\exp\left(\frac{ix^2}{2t}\right),
\end{aligned} \tag{189}$$

and similarly for v and w (though of course we can use a shortcut for v when remembering it's the complex conjugate of u, and that $\overline{H_1} = H_2$). This therefore lets us rewrite $H_A = 0$ as

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial y^1}\left(\frac{\partial w^u}{\partial y^1}\right) + \frac{\partial}{\partial y^2}\left(\frac{\partial w^u}{\partial y^2}\right) + i\frac{\partial w^u}{\partial y^3} + w^u\left(\frac{i}{y^3} - w^w\right) = 0, \\
G_2 &= \frac{\partial}{\partial y^1}\left(\frac{\partial w^v}{\partial y^1}\right) + \frac{\partial}{\partial y^2}\left(\frac{\partial w^v}{\partial y^2}\right) - i\frac{\partial w^v}{\partial y^3} - w^v\left(\frac{i}{y^3} + w^w\right) = 0, \\
G_3 &= -\left(\frac{1}{y^3}\right)^2 + \frac{\partial}{\partial y^1}\left(\frac{\partial w^w}{\partial y^1}\right) + \frac{\partial}{\partial y^2}\left(\frac{\partial w^w}{\partial y^2}\right) - w^u w^v = 0.
\end{aligned} \tag{190}$$

All this effort just to reduce the equations by one variable!

To proceed further, we will need symmetries of these equations; but note that $\mathbf{X_4}$, a rotation in the $y-z$ plane, commutes with $\mathbf{X_9}$ (with $\mathbf{T} = (t^2, 0, 0)$). It is thus an inherited symmetry of (190), and in our new co-ordinates is equivalent to

$$\mathbf{X_4} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} = y^2\frac{\partial}{\partial y^1} - y^1\frac{\partial}{\partial y^2}. \tag{191}$$

Infinitesimal point transformations give $(y^1)^2 + (y^2)^2$ to be constant under such a transformation. So take $a = y^3, b = (y^1)^2 + (y^2)^2$ to be independent variables (since they are invariant under $\mathbf{X_4}$) that make a solution $w^\alpha(a, b)$, and apply the chain rule:

$$\frac{\partial w^\alpha}{\partial y^1} = \frac{\partial w^\alpha}{\partial a}\frac{\partial a}{\partial y^1} + \frac{\partial w^\alpha}{\partial b}\frac{\partial b}{\partial y^1} = 2y^1\frac{\partial w^\alpha}{\partial b},$$

$$\begin{aligned}
\frac{\partial}{\partial y^1}\left(\frac{\partial w^\alpha}{\partial y^1}\right) &= 2\frac{\partial w^\alpha}{\partial b} + 4(y^1)^2\frac{\partial^2 w^\alpha}{\partial b^2}, \\
\frac{\partial}{\partial y^2}\left(\frac{\partial w^\alpha}{\partial y^2}\right) &= 2\frac{\partial w^\alpha}{\partial b} + 4(y^2)^2\frac{\partial^2 w^\alpha}{\partial b^2}, \\
\frac{\partial w^\alpha}{\partial y^3} &= \frac{\partial w^\alpha}{\partial a},
\end{aligned} \tag{192}$$

which again reduces the SN equations to

$$\begin{aligned}
F_1 &= 4\frac{\partial w^u}{\partial b} + 4b\frac{\partial^2 w^u}{\partial b^2} + i\frac{\partial w^u}{\partial a} + w^u\left(\frac{i}{a} - w^w\right) = 0, \\
F_2 &= 4\frac{\partial w^v}{\partial b} + 4b\frac{\partial^2 w^v}{\partial b^2} - i\frac{\partial w^v}{\partial a} - w^v\left(\frac{i}{a} + w^w\right) = 0, \\
F_3 &= 4\frac{\partial w^w}{\partial b} + 4b\frac{\partial^2 w^w}{\partial b^2} - \frac{1}{a^2} - w^u w^v = 0.
\end{aligned} \tag{193}$$

To reduce further to just one variable, we would need a symmetry that commutes with both $\mathbf{X_4}$ and $\mathbf{X_9}$; $\mathbf{X_{10}}$ is the last remaining generator which does so. Writing

$$\mathbf{X_{10}} = (\mathbf{X_{10}}a)\frac{\partial}{\partial a} + (\mathbf{X_{10}}b)\frac{\partial}{\partial b} + (\mathbf{X_{10}}w^\alpha)\frac{\partial}{\partial w^\alpha} \tag{194}$$

and inserting the identities (188) gives

$$\mathbf{X_{10}} = i\Omega(a)\left[w^u\frac{\partial}{\partial w^u} - w^v\frac{\partial}{\partial w^v}\right] - \Omega'(a)\frac{\partial}{\partial w^w}, \tag{195}$$

which does not help us to reduce the SN equations any further (this may be due to the rotational symmetry we used; if we were working in polar co-ordinates, it would 'knock out' two independent variables at once).

3.4.3 Generators: $\mathbf{X_{10}}$

Finally, $\mathbf{X_{10}}$ just gives us a time evolution of the solutions:

$$\tilde{x}^i = x^i, \qquad \tilde{u} = ue^{i\varepsilon\Omega(t)}, \qquad \tilde{w} = w - \varepsilon\Omega'(t). \tag{196}$$

That is, changing the phase of ψ will alter ϕ; depending on the sign of ϕ and our choice of an arbitrary $\Omega(t)$, a sufficiently large t may give $\phi = 0$ at certain points. When applied to (non-dimensionalised) stationary states, this represents a transformation

$$\tilde{\Psi}(\mathbf{x},t) = \Psi(\mathbf{x},t)e^{i\varepsilon\Omega(t)} = \psi(\mathbf{x})e^{-i[Et-\varepsilon\Omega(t)]}. \tag{197}$$

3.5 Symmetries of the time-independent SN equations

To calculate the time-independent symmetries, one may at first consider making a 'transformation' $i\frac{\partial}{\partial t} \to E$ as is often the case in quantum mechanics, or even just calculate them again, starting from equations (138).

However, a simpler method is to simply set $t \equiv 0$ and take $w \mapsto w - E$; doing such a thing transforms $H_A = 0$ into the system

$$\begin{aligned} G_1 &= u_{,11} + u_{,22} + u_{,33} - uw + Eu = 0, \\ G_2 &= v_{,11} + v_{,22} + v_{,33} - vw + Ev = 0, \\ G_3 &= w_{,11} + w_{,22} + w_{,33} - uv = 0 \end{aligned} \tag{198}$$

which we recognise to be the time-independent SN equations. Ignoring t also means we can ignore the arbitrary time functions that appear through integration. Also note that we are justified in doing this since

$$\eta^w = \left.\frac{\partial(\tilde{w} - E)}{\partial\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial\tilde{w}}{\partial\varepsilon}\right|_{\varepsilon=0} \tag{199}$$

gives the same transformations. Thus, the generators (179) turn into

$$\begin{aligned} \mathbf{W_1} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} - 2(w-E)\frac{\partial}{\partial w}, \\ \mathbf{W_2} &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\ \mathbf{W_3} &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \\ \mathbf{W_4} &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\ \mathbf{W_5} &= \frac{\partial}{\partial x}, \qquad \mathbf{W_6} = \frac{\partial}{\partial x}, \qquad \mathbf{W_7} = \frac{\partial}{\partial x}. \end{aligned} \tag{200}$$

(Given the 'shortcut' of starting from equations (138), checking this does not take long at all.) These are simple transformations that we've seen before and would expect, including $\mathbf{W_1}$, which corresponds to the scalings

$$\tilde{x}^j = x^j e^{\varepsilon}, \qquad \tilde{u} = u e^{-2\varepsilon}, \qquad \tilde{w} - E = (w - E) e^{-2\varepsilon}. \tag{201}$$

4 Conclusions

We have shown that the time-dependent SN equations admit several Lie point symmetries: rotations, translations, scalings, and a a 'pseudo-Galilean' transformation (with the first three of these appearing in the symmetries of the time-independent equations). This particular transformation shows that, in a moving frame, ψ will undergo a phase shift, and the self-potential ϕ will change only in an accelerating frame (184), which seems somewhat analogous to the equivalence principle of general relativity. A reduction of the equations into two independent variables was also carried out, but time did not allow a full reduction.

But do they give the 'basis' of collapsing/collapsed states that we are after? Numerical work on the SN equations indicates that, in the spherically symmetric time-dependent case, the ground state is the only stable solution [17, 19], and that higher excited states decay into a 'multiple' of the ground state with normalisation constant less than 1 – this would mean that the time-dependent SN equations are *not* giving us the basis states for collapse (since such states should have a 'full' normalisation). Non-dimensionalising the equations to fit with given mass and length scales didn't lead us to scalings that fit with proposed time-frames for collapse either, though it was never proposed that such values could be derived from the SN equations.

A better model for gravitationally induced wavefunction collapse is still required regardless, assuming that we don't really live in a multiverse and Afshar's experiment does contain a fatal flaw, and many of the arguements presented need shoring up. However, one should hope to see experimental results that confirm or deny such a thing within a few years, and perhaps there will even be some support from the various studies into quantum gravity. As ever with physics, it should be much easier to proceed with real data in hand.

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